Brief paper

Robust learning-based MPC for nonlinear constrained systems

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**ABSTRACT**

This paper presents a robust learning-based predictive control strategy for nonlinear systems subject to both input and output constraints, under the assumption that the model function is not known a priori and only input–output data are available. The proposed controller is obtained using a nonparametric machine learning technique to estimate a prediction model. Based on this prediction model, a novel stabilizing robust predictive controller without terminal constraint is proposed. The design procedure is purely based on data and avoids the estimation of any robust invariant set, which is in general a hard task. The resulting controller has been validated in a simulated case study.

1. Introduction

Model-based control design, and particularly model predictive control (MPC), relies on the availability of an accurate description of the plant. When a model of the plant dynamics is unavailable a priori, system identification methods can be employed to devise such models automatically from observational data. The objective of this paper is to design a predictive controller based on such a learning method. In this setting, the learning method should be flexible enough to learn rich classes of dynamical systems, while at the same time, it should offer bounds on its predictive performance. The latter is important in the predictive control setting if one wishes to give guarantees on the performance and feasibility of the data-based controller, e.g. for nonlinear (Allgöwer & Zheng, 2012) or cyber–physical systems (Behl, Jain, & Mangharam, 2016).

Learning and data-driven predictive controllers have recently gained the attention of the control community (Hewing, Wabersich, Menner, & Zeilinger, 2019). An approach to learning-based MPC that is independent of the concrete learning paradigm was proposed in Aswani, Gonzalez, Sastry, and Tomlin (2013). A broad scope some research consider direct weight optimization methods (Piga, Formentin, & Bemporad, 2017; Salvador, Ramirez, Alamo, & Muñoz de la Peña, 2019), others Gaussian processes (Fisac et al., 2018; Maiworm, Limon, Manzano, & Findeisen, 2018), or random forests (Smarra et al., 2018), among many others. Several previous works on this topic have used nonlinear set membership (NSM) methods (Canale, Fagiano, & Signorile, 2014) for learning, like (Milanese & Novara, 2004; Sukharev, 1978). In previous works, the authors proposed to use kinky inference methods (Calliess, 2014) in a model predictive control setting. Kinky inference (KI) methods encompass Lipschitz interpolation (Beliakov, 2006; Sukharev, 1978) and NSM methods (Milanese & Novara, 2004), and they have several properties that are interesting for MPC. A modified version of this method, tailored to model predictive control, was proposed and used to design a MPC with guaranteed closed-loop properties for systems subject to input constraints (Manzano, Limon, Muñoz de la Peña, & Calliess, 2019). The main limitation of this controller is that it cannot guarantee robust satisfaction of output constraints.

In this paper, the main contribution is a new robust predictive controller for systems that also have output constraints. A design method that takes into account the prediction error bounds in an explicit way to tighten the problem constraints and that guarantees closed-loop constraint satisfaction and input-to-state stability (ISS) (Limon et al., 2008) is provided. One of the main characteristics of this design is that it is not based on a terminal region constraint. In general, this terminal constraint is based on robust invariant sets which, for the class of systems considered (that is, unknown systems, possibly nonlinear, for which a priori only input/output data is available), are difficult to obtain.

Another contribution of this work is the procedure to define the tightened constraints of the MPC optimization problem. They
are specifically tailored to the inference model used, in order to obtain the least conservative possible bounds. In contrast to the preliminary version of this controller, presented in Manzano et al. (2018), an output-feedback formulation is considered, and the stability analysis is addressed.

**Notation:** For two column vectors $v, w, (v, w)$ implies $[v^T, w^T]^T$. Given two sets $A, B, A \oplus B$ is the Minkowski sum and $A \ominus B$ the Pontryagin difference. The set of the integers in the interval $[a, b]$ is denoted $\mathbb{Z}_{a,b}$. A function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a $K$-function if it is continuous, strictly increasing and $\alpha(0) = 0$. Given a vector $v \in \mathbb{R}^n$, the ball $B(v) \subset \mathbb{R}^n$ is defined as $B(v) = \{y : |y| \leq v, \ y \in \mathbb{R}^n\}$; and $|v|$ denotes the vector whose components are the absolute value of the components of $v$. $I_n$ is an identity matrix of size $n$. Given a compact set $\Omega$, $\|\cdot\|_\Omega$ for some norm $\|\cdot\|$.

**2. Problem setting**

In this paper it is assumed that the system to be controlled is a sampled continuous-time system described by an a priori unknown discrete-time model, where $y(k)$ in $\mathbb{R}^n$ is its measured output and $u(k) \in \mathbb{R}^n$ is the control input. Both inputs and outputs are subject to hard constraints

$$u(k) \in \mathcal{U}, \quad y(k) \in \mathcal{Y},$$

where both $\mathcal{U}$ and $\mathcal{Y}$ are compact sets. It is assumed, without loss of generality, that the origin is the equilibrium point of the system where the plant must be stabilized.

It is assumed that the only information available from the plant is historical data, containing a certain set of measured inputs and outputs trajectories, $\mathcal{D}$. The objective of the paper is to design an output-feedback control law

$$u(k) = \kappa(y(k); \mathcal{D}),$$

such that from the data set $\mathcal{D}$ and the current output measurement $y(k)$, the control action is computed. It is desired to devise the control law such that the closed-loop system is asymptotically stable and that the constraints are satisfied for all time steps $k \in \mathbb{N}$.

Since a model of the dynamics is not available a priori, it is assumed that the measured output can be used to describe the model of the system with the following nonlinear autoregressive exogenous (NARX) model of the plant (Leontaritis & Billings, 1985; Levin & Narendra, 1997):

$$y(k + 1) = f(x(k), u(k)) + e(k),$$

where $x(k) = (y(k), \ldots, y(k - n_y), u(k - 1), \ldots, u(k - n_u)) \in \mathcal{X} := \mathbb{R}^{(n_y + 1) \times (n_u + 1)}$ with $n_y = (n + 1)n_y + n_u n_u$, for some memory horizon lengths $n_y, n_u \in \mathbb{N}$. The residual $e(k)$ models process noise and it is assumed to be confined to a compact set $\mathcal{E} \subset \mathbb{R}^n$. For notational convenience, the inputs of $f$ are aggregated into a joint vector $w := (x, u) \in \mathcal{W}$, which is referred to as regressor.

**Remark 1.** The horizons $n_u$ and $n_y$ represent the model order. In Levin and Narendra (1997), the conditions under which they could be taken as $n_u = n_y = 2n$ were given, where $n$ is the order of the system. If $n$ is unknown, one would have to use the best guess, or cross-validation methods to estimate $n_u$ and $n_y$.

**Assumption 1 (Hölder Continuity).** Each output component of the function $f(\cdot)$, referred to as ground truth function, is Hölder continuous. That is, there exist some constants $L_f, \ p_f \in (0, 1] (i \in \mathbb{I}_1^p)$ such that $\forall w_1, w_2 \in \mathcal{W}$

$$|f_i(w_1) - f_i(w_2)| \leq L_f, \|w_1 - w_2\|_\mathcal{W}^{p_f},$$

where $\|\cdot\|_\mathcal{W}$ stands for a specific norm defined for the regressors, and the sub-index $i$ denotes the $i$th component of the vector. For each $i \in \mathbb{I}_1^p$, any constant $L_f, i$ that satisfies this condition is called a Hölder constant, while the lowest of them is called the best Hölder constant, $L_f, i$.

**Remark 2.** Assumption 1 can be relaxed to general continuity, provided that both $\mathcal{Y}$ and $\mathcal{U}$ are compact sets.

**2.1. The learning method**

In this section the machine learning method used to estimate $f$ is presented, sometimes called kinkify inference (Calliess, 2014). Using the available experimental data, a data set $\mathcal{D}$ of $N_D$ regressor/outputs is collected; that is

$$\mathcal{D} := \{(y(j), w(j)) \mid j = 1, \ldots, N_D\}.$$  

The structure of $\mathcal{D}$ depends on the value of $n_u$ and $n_y$. To predict an unseen query point $w$, IMK makes use of the data base $\mathcal{D}$, and provides an estimation of the Hölder parameters, denoted $L$ and $p$, which are vectors of dimension $n_y$.

**Definition 1 (Kinky Inference Rule).** The $i$th output component function of the KL predictor, for $i = 1, \ldots, n_y$, shall be defined by

$$\hat{y}(k + 1) = \hat{f}(x(k), u(k)).$$

This predictor $\hat{f}$ is Hölder continuous, and as proven in Calliess (2016, Lemma 5), the Hölder parameters of $f$ are also Hölder parameters of $\hat{f}$. That is, for a given exponent $p$, the Lipschitz constant $L^*$ is also a Lipschitz constant of $\hat{f}$. However, the estimated $L$ and $p$ will suffice to derive the stability properties of the proposed controller, as it will be explained later.

**Remark 3.** While in several works a priori knowledge of the correct parameters $L_f$ and $p_f$ is assumed (Canale et al., 2014; Zabin et al., 2003), other works provide methods of calculating these parameters from the available data (Calliess, 2016; Milanese & Novara, 2004). In this paper, the so-called LACKI method (Calliess, 2016) is applied. It obtains the Lipschitz constant $L$ as the minimum one that is consistent with the data. The learning feature of the proposed predictor is also demonstrated in Calliess (2016). Note that, w.l.o.g., this paper extends the LACKI method to each output component function in isolation.

In this paper, the paradigm of predictive control will be employed to derive the data-based control law (2). MPC requires repeated optimization of the predicted control inputs subject to constraints. Therefore, in order to give guarantees on the controller’s closed-loop performance, recursive feasibility and constraint satisfaction must be ensured. That is, it is necessary to ensure that all constraints remain satisfiable during runtime, or equivalently, to guarantee that the controlled system will not

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1 In order to reduce the estimation error, different Hölder constants $L_f$ are used to estimate the $i$th entry of the output vector. With a slight abuse of notation $L$ and $p$ will denote a vector in $\mathbb{R}^p$.

2 For the sake of conciseness, the dependence of $\hat{f}$ with $L$, $p$ and $\mathcal{D}$ may be omitted in the rest of the paper.
leave the feasibility region. However, since the controller will not
to be based on the ground-truth dynamics $f$, but on the learned
model $\hat{f}$ inferred from a sample of the ground-truth, recursive
feasibility can only be guaranteed if a bound on the discrepancy
between $f$ and $\hat{f}$ is known a priori and taken into account by the
controller.

The estimation method ensures that if the model function is
Hölder and the noise is bounded, then the estimation error is
bounded (Calliess, 2016), which is required to design a deter-
mimistic robust controller to regulate the plant. Any worst-case
guarantee inevitably requires a priori knowledge. Hence, in the
following hypothesis, it is assumed that this bound is available
for the design of the controller.

**Assumption 2.** It is assumed that for $L, p$, and $D$, a bound on
the error between the estimated output and the real output is known,
denoted $\mu \in \mathbb{R}^{n_u}$, such that

$$
|\hat{f}(x, u) - f(x, u) - e_i| \leq \mu_i,
$$

(8)

for all $i \in \mathbb{N}^n$, $e \in \mathcal{E}$, $x \in \mathcal{Y}^{n_u+1} \times \mathcal{U}^{n_u}$, and $u \in \mathcal{U}$.

**Remark 4.** From a practical point of view the problem of how
to calculate the error bound must be addressed. Kinley inference
methods enjoy the property of providing a deterministic error
bound if the Lipschitz constant and an upper bound of the noise
are known (Calliess, 2014). Moreover, if a bound on the second
derivative is known, it is also possible to derive an estimation
error bound. If these parameters are not known a priori, which
is usual in practice, then they must be estimated from experi-
mental data. Consequently, the validity of the results presented in
this paper is conditioned to the validity of the estimated error
bound. This is the reason why this is considered as a standing
assumption.

**Remark 5.** The K1 prediction method has recently been improved
in Manzano et al. (2019) decreasing the computational cost and
smoothing the prediction, in order to enhance the optimization
that will be carried out by the controller.

3. Stabilizing data-based NMPC

In this section, a model predictive controller is derived based
on a prediction model learned from data of the plant. Since
the prediction model is not accurate, the effect of the estimation error
on the predictions must be analysed to be taken into account
in the design of the controller. For this analysis, it is convenient
to define the NARX model of the plant in a state–space form as
follows:

$$
x(k + 1) = F(x(k), u(k)) + \xi(k)
$$

(9a)

$$
y(k) = Mx(k),
$$

(9b)

where

$$
F(x(k), u(k)) = \left( f(x(k), u(k)), y(k), \ldots, y(k - n_u + 1), \right.
$$

(10)

$$
\ldots, u(k - n_b + 1) \right)
$$

$$
M = [I_{n_y}, 0, \ldots, 0], \text{ and } \xi(k) = (\epsilon(k), 0, \ldots, 0).
$$

Let $\hat{y}(j|k)$ denote the output that, at time $k$, is predicted to be
observed at time $k + j$, for a given candidate control sequence
$u(k + j), j \in \mathbb{N}^{n_b}$. Then the predicted state is given by

$$
\hat{x}(j|k) = \hat{F}(\hat{x}(j|k), u(k + j)),
$$

where

$$
\hat{x}(j + 1|k) = \hat{F} (\hat{x}(j|k), u(k + j))
$$

(11)

and

$$
\hat{F}(\hat{x}(j|k), u(k + j)) = \left( \hat{\tilde{f}}(\hat{x}(j|k), u(k + j)), \right.
$$

$$
\hat{\tilde{y}}(j|k), \ldots, \hat{\tilde{y}}(j|k), y(k + j - n_u + 1), \ldots
$$

$$
\ldots, u(k + j), \ldots, u(k + j - n_b + 1) \right).
$$

The proposed robust MPC is based on nominal predictions and
tightened constraints. To guarantee robustness, a bound on the
propagation of the prediction error (see Fig. 1) is calculated from
the following lemma:

**Lemma 1.** Assume that at sampling time $k$, the state of the plant
is $x(k)$ and a sequence of future control inputs $u(k + j)$, $j \in \mathbb{N}^{n_b}$
is given. Let $\hat{x}(j|k)$ and $\hat{y}(j|k)$ be the predicted states and outputs,
respectively, derived from (11) for the given sequence of future
control inputs and the current state $x(k)$, i.e., $\hat{x}(0|k) = x(k)$.

Assume that at sampling time $k + 1$, the current output $y(k + 1)$
is measured, and hence the current state $x(k + 1)$ is known. Based
on these new measurements, an updated sequence of states and
outputs $\check{x}(j|k + 1)$ and $\check{y}(j|k + 1)$ is predicted based on (11) with
$\check{x}(0|k + 1) = x(k + 1)$ and the remaining sequence of the given
future control inputs.

Let $c_1 \in \mathbb{R}^{n_y}$ be a vector such that

$$
|y(k + 1) - \check{y}(j|k + 1)| \leq c_1.
$$

Then, the mismatch between the predictions satisfies

$$
|\hat{y}(j - 1|k + 1) - \hat{y}(j|k)| \leq c_j, j \in \mathbb{N}^{n_b},
$$

(13a)

$$
|\check{y}(j - 1|k + 1) - \check{y}(j|k)| \leq r_j, j \in \mathbb{N}^{n_b},
$$

(13b)

where $c_j \in \mathbb{R}^{n_y}$ and $r_j \in \mathbb{R}$ are obtained from the recursion

$$
c_{j+1,i} = L_i r_p^{n_b},
$$

(14)

and $r_i = \|\mathbf{z}_i\|_\pi, j \in \mathbb{N}^{n_b}, i \in \mathbb{N}^{n_b}$, where

$$
\mathbf{z}_i = \mathcal{B}(c_j) \times \cdots \times \mathcal{B}(c_j) \times \{0\} \times \cdots \times \{0\} \subseteq \mathbb{R}^{n_y},
$$

with $\sigma(j) = \max(1, j - n_b)$.

**Proof.** Provided that $\hat{y}(j - 1|k + 1) = \hat{\tilde{f}}(\check{x}(j - 2|k + 1), u(k + j - 1))$
and $\hat{y}(j|k) = \hat{\tilde{f}}(\check{x}(j - 1|k), u(k + j - 1))$, it can be derived that $\forall i \in \mathbb{N}^{n_b}$

$$
|\hat{y}(j - 1|k + 1) - \hat{y}(j|k)| \leq L_i \|\check{x}(j - 2|k + 1) - \check{x}(j - 1|k)\|_\pi.
$$

Given that

$$
\check{x}(j - 2|k + 1) - \check{x}(j - 1|k) =
$$

$$
\left[ \check{y}(j - 2|k + 1) - \check{y}(j - 1|k), \check{y}(j - 3|k + 1) - \check{y}(j - 2|k),
\right.
$$

$$
\ldots, \check{y}(\sigma(j - 1|k + 1) - 1|k) - \check{y}(\sigma(j - 1|k + 1)|k) \ldots, 0 \right].
$$

3 $\| \cdot \|_\pi$ is a norm for the state–space such that $\| x \|_\pi = \| (x, 0) \|_W$. 

![Fig. 1. Propagation of the prediction error.](image-url)
then $\tilde{x}(j-2|k+1) - \tilde{x}(j-1|k) \in \mathcal{S}_{j-1}$. Assuming that $c_{j-1}$ is known, 
\[ \|\tilde{x}(j-2|k+1) - \tilde{x}(j-1|k)\|_\mathcal{x} \leq \|\mathcal{S}_{j-1}\|_\mathcal{x} = r_{j-1}, \]
which implies the stated result. \[ \square \]

**Remark 6.** If the infinity norm is chosen as the norm of the input space then
\[ r_j = \max_{s \in [u]} \|c_s\|_\infty. \]

Based on the derived bounds on the prediction error, the problem of robust constraint satisfaction is addressed by means of a set of tightened constraints on the outputs (Rawlings & Mayne, 2009), computed offline for the maximum possible prediction error, i.e. taking $c_1 = \mu$. These sets are defined as follows:
\[ \mathcal{Y}_j = \mathcal{Y} \ominus \mathcal{B}(d_j), \]
(15)
where
\[ d_j = \sum_{i=1}^j c_i. \]
(16)
These constraints sets will be used to prove recursive feasibility of the controller, following standard procedures.

**Lemma 2.** The sets $\mathcal{Y}_j$ are such that for all $y \in \mathcal{Y}_j$ and for all $\Delta y \in \mathcal{B}(c_j)$, $y + \Delta y \in \mathcal{Y}_{j-1}$.

**Proof.** Since for $j \geq 1$, $d_j = d_{j-1} + c_j$, it follows that $\mathcal{B}(d_j) = \mathcal{B}(d_{j-1}) \oplus \mathcal{B}(c_j)$.
By definition,
\[ y + \Delta y \in \mathcal{Y}_j \ominus \mathcal{B}(c_j) = \mathcal{Y} \ominus \mathcal{B}(d_j) \ominus \mathcal{B}(c_j), \]
and hence $\mathcal{Y}_j = \mathcal{Y} \ominus \mathcal{B}(d_j) = \mathcal{Y} \ominus \mathcal{B}(d_{j-1}) \ominus \mathcal{B}(c_j)$, so
\[ y + \Delta y \in \mathcal{Y}_j \ominus \mathcal{B}(c_j) \]
\[ = \mathcal{Y} \ominus \mathcal{B}(d_{j-1}) \ominus \mathcal{B}(c_j) \ominus \mathcal{B}(c_j) \]
\[ \subseteq \mathcal{Y} \ominus \mathcal{B}(d_{j-1}) = \mathcal{Y}_{j-1}. \]
\[ \square \]

In order to ensure that the proposed controller is feasible, the tightened set of constraints must be non-empty along the prediction horizon, as stated in the following assumption:

**Assumption 3.** The prediction horizon $N$ and the estimation error bound $\mu$ are such that the set $\mathcal{Y}_N$ is non-empty.

Based on the previous definitions, the optimization problem $P_N(\hat{x}(k); \mathcal{D})$ of the proposed predictive controller is:
\[ \min_{u} V_N(\hat{x}(k), u) \]
\[ = \sum_{i=0}^{N-1} \ell(\hat{x}(i|k), u(i)) + \lambda \mathcal{V}_n(\hat{x}(N|k)) \]
(17a)
s.t. $\hat{x}(0|k) = \hat{x}(k)$
(17b)
$\hat{x}(j+1|k) = \hat{F}(\hat{x}(j|k), u(j)), j \in \mathbb{N}_{N-1}$
(17c)
$\hat{y}(j|k) = \hat{M}(\hat{x}(j|k), j \in \mathbb{N}_{N-1}$
(17d)
$u(j) \in \mathcal{U}, j \in \mathbb{N}_{N-1}$
(17e)
$\hat{y}(j|k) \in \mathcal{Y}_j, j \in \mathbb{N}_{N-1}$
(17f)
where $\lambda \geq 1$ is a weighting parameter. Note that this problem is non-linear, non-convex and non-differentiable. Its ingredients are required to meet the following assumption, which is similar to the standard MPC ones (Rawlings & Mayne, 2009):

**Assumption 4.**

1. The stage cost function $\ell(\cdot, \cdot)$ is a Hölder continuous positive definite function such that $\ell(\cdot, \cdot) \geq \alpha(\|\mathcal{x}\|_\mathcal{x})$ for a certain $\alpha$-function $\alpha$, and its Hölder parameters are $\phi$ and $p$.
2. There exists a control law $u = k(\hat{x})$, a function $V_f$ and a level set $\Omega = \{x : V_f(x) \leq \gamma\} \subseteq \mathbb{R}^n$ for some $\gamma > 0$ such that for all $x \in \Omega$, the following conditions hold:
   a) $V_f$ is a Hölder continuous positive definite function, with Hölder constants $L_V, \phi_V$, such that
   \[ \alpha(\|\mathcal{x}\|_\mathcal{x}) \leq V_f(\hat{x}(N|k)) \leq \beta(\|\mathcal{x}\|_\mathcal{x}). \]
   \[ V_f(\hat{x}(N, k|k)) - V_f(x) \leq -\ell(\hat{x}, k|k). \]
   b) $k(\hat{x}) \in \mathcal{U}$, $\mathcal{M}x \in \mathcal{Y}_N$.

The controller is derived from the receding horizon solution of (17). It follows a standard robust approach in which the cost of the nominal predictions is minimized, while taking into account a tightened set of constraints to guarantee recursive feasibility. The main difference with off-the-shelf robust ISS formulations for nonlinear systems (Limone et al., 2008) is that in these, either there are no constraints on the states in the optimization problem, or a terminal constraint, based on a certain robust positive invariant set, is added. In this controller, although a terminal cost (based on a local controller for the nominal model) is taken into account in the cost function, no terminal constraint is included. Thus, its design is notably simplified since the calculation of a robust invariant set is avoided, which was a hard task, as shown in Fiaccini, Alamo, and Camacho (2010). In this case, the calculation could have been even more difficult provided the lack of an explicit expression of the model of the system.

Furthermore, an additional tuning parameter $\lambda$ is added, modifying the weight of the terminal cost in the objective function. It is proven that this controller guarantees that the closed-loop system is ISS in an explicitly defined region of the state space, which is enlarged by this weight.

Define the function
\[ v_1 = \sum_{j=1}^{N} L_V r_{j}^p + \lambda \mathcal{L}_N \mathcal{P}_{N-1}, \]
where $r_j$ is defined in Lemma 1 for $c_1$, and $L_V, \mathcal{L}_N$ in Assumption 4.

**Assumption 5.** The bound $\mu$ is such that the set $\mathcal{Y} = \{x : \ell(x, 0) \leq v_1(\mu)\}$ is contained in $\Omega$. The positive constants $\phi$ and $\phi_1$ are such that $\lambda \geq 1$ and $\ell(x, 0) > \phi$ for all $x \notin \Omega$.

**Remark 7.** In a general setting, a condition to check if the level set $\mathcal{Y}$ is contained in $\Omega$, could be derived using the supply $\mathcal{K}$-functions that bound the cost functions given in Assumption 4. In this case the condition would be:
\[ v_1(\mu) \leq \alpha_1(\beta_1^{-1}(\gamma)). \]
(18)
Another method could be using probabilistic validation by means of randomized algorithms (Calafiore & Dabbene, 2006).

**Lemma 3.** Under Assumption 5, $\phi \geq v(\mu)$.

**Proof.** Since $\mathcal{Y} \subseteq \Omega$, the constant $\phi$ can be taken as $\phi = \min_{x \in \mathcal{X}, \gamma \in \mathcal{Y}} \ell(x, 0)$.

Let $\Gamma$ define the following level set of the optimal cost function
\[ \Gamma = \{x : V^\mu_N(x) \leq N \phi + \lambda \gamma\}. \]
It is next proven that this set defines the region in which ISS is guaranteed. Notice that this set is compact and non-empty.

**Theorem 1 (ISS Stability).** Suppose that Assumptions 2–5 hold for the optimization problem $P_0(\cdot)$. Let $\kappa_N(x)$ be the control law derived from the solution of $P_0(x; D)$ applied using a receding horizon policy. Then, for any $x(0) \in \Gamma$, the system controlled by the control law $u(k) = \kappa_N(x(k))$ is input-to-state stable w.r.t. the estimation error, and the constraints are always satisfied, i.e., $u(k) \in U$, $y(k) \in \gamma$, and $x(k) \in \Gamma$, $\forall k$.

**Proof.** Assume that $x(k) \in \Gamma$. Then, it can be defined that $x^*(N|k) \in \Omega_y$ (Limon, Alamo, Salas, & Camacho, 2006). Define the shifted sequence as $\tilde{u}(j + 1)$ such that $\tilde{u}(j + 1) = u^r(j + 1)$ for $j \in I_k^{N - 1}$ and $\tilde{u}(N - 1 + 1) = \kappa_N(x(N|k))$.

Recursive feasibility: Assuming that $x(k) \in \Gamma$, it will be proven that $x(k + 1) \in \Gamma$. Since $\Gamma$ is a subset of the feasibility region of the optimization problem $P_0(x; D)$, the system is recursively feasible.

Firstly, it will be shown that the solution $\tilde{u}(k + 1)$ is a feasible solution for $x(k + 1)$. Given that $x^*(N|k) \in \Omega_y$, from the feasibility of $u^r$ and Assumption 4, it is immediate to state that $\tilde{u}(j + 1) = u^r(j + 1)$ for all $j \in I_k^{N - 1}$.

From Lemma 1, $\tilde{y}(j + 1) - y^*(j + 1) \in B_{C_1}$, $\forall j \in I_k^{N - 1}$, and from the feasibility of $u^r$. Thus, $y^*(s|k) \in \gamma_1$, for $s \in I_k^{N - 1}$ and $x^*(N|k) \in \Omega_y$, which implies that $y^*(N|k) = M x^*(N|k) \in \gamma_N$ in virtue of Assumption 4.2b. Then, from Lemma 2, for all $j \in I_k^{N - 1}$, $\tilde{y}(j + 1) = \gamma_1 \Rightarrow B_{C_1} \subseteq \gamma_1$. Therefore the problem $P_0(x(k + 1); D)$ is feasible.

Next, it will be proven that $x(k + 1) \in \Gamma$. Since $x(k) \in \Gamma$, $V_{\gamma}(x(k + 1)) \leq N \phi + \kappa_N y$. Following standard arguments in MPC (Rawlings & Mayne, 2009) it can be proven that

$$V_{\gamma}(x^*(1|k), \tilde{u}(k + 1)) \leq V_{\gamma}(x(k)) - \tilde{\xi}(x(k), u(k)) \leq N \phi + \kappa_N y - \tilde{\xi}(x(k), u(k)).$$

On the other hand, given $x = \bar{F}(x^*(N|k), \kappa_N(x^*(N|k)))$, $V_{\gamma}(x(k + 1), \tilde{u}(k + 1)) = V_{\gamma}(x^*(1|k), \tilde{u}(k + 1))$

$$\sum_{i=0}^{N-1} \left( \tilde{\xi}(i|k + 1), \tilde{u}(i|k + 1) \right) - \tilde{\xi}(x^*(i + 1|k), \tilde{u}(i|k + 1)) + \lambda \left( V_{\gamma}(x^*(N|k), \tilde{u}(N)) - V_{\gamma}(x) \right).$$

According to Lemma 1, it can be derived that for $s \in \mathbb{N}^x_1$, $\|\bar{F}(s - 1|k + 1) - x(s|k)\|_x \leq r_s$, with $r_s$ obtained for a given $c_1$. Then, for $j \in I_k^N$, and given $c_1$ satisfying (12),

$$\tilde{\xi}(j|k + 1), \tilde{u}(j|k + 1) \leq L_{\xi} r_{j + 1}^{P_{\xi}}$$

and $V_{\gamma}(x^*(N|k)) = V_{\gamma}(x) \leq L_{\xi} r_{j + 1}^{P_{\xi}}$. Therefore,

$$V_{\gamma}(x^*(k + 1), \tilde{u}(k + 1)) - V_{\gamma}(x^*(1|k), \tilde{u}(k + 1)) \leq \nu(c_1).$$

To prove robust invariance the worst possible case has to be considered, for which $c_1 = \mu$ is taken. Hence, it has been proven that

$$V_{\gamma}(x(k + 1), \tilde{u}(k + 1)) \leq \nu(\mu) + N \phi + \kappa_N y - \tilde{\xi}(x(k), u(k)).$$

Consider the case where $x(k) \in \Gamma \setminus \gamma$. Then $\tilde{\xi}(x(k), u(k)) > \nu(\mu)$. Hence, $V_{\gamma}(x(k + 1), \tilde{u}(k + 1)) \leq \nu(\mu) + N \phi + \kappa_N y - \tilde{\xi}(x(k), u(k)) \leq N \phi + \kappa_N y$.

Given that $V_{\gamma}^N(x(k + 1)) \leq V_{\gamma}(x(k + 1), \tilde{u}(k + 1))$ then $x(k + 1) \in \Gamma$.

Consider now the case that $x(k) \in \gamma$. Since $\gamma \subseteq \Omega_y$, $x(k) \in \Omega_y$. With standard arguments in MPC (Rawlings & Mayne, 2009), it can be shown that $V_{\gamma}^N(x(k)) \leq V_{\gamma}(x(k)) \leq \lambda \gamma$. Hence,

$$V_{\gamma}(x(k + 1), \tilde{u}(k + 1)) \leq \nu(\mu) + V_{\gamma}^N(x(k)) - \tilde{\xi}(x(k), u(k)) \leq \nu(\mu) + \kappa_N y - \tilde{\xi}(x(k), u(k)),$$

since $\nu(\mu) \leq \phi$, $V_{\gamma}(x(k + 1), \tilde{u}(k + 1)) \leq N \phi + \kappa_N y$. Thus, $x(k + 1) \in \Gamma$.

**Input-to-state stability:** Eq. (19) can be rewritten as follows, taking $c_i = e_i(k + 1) := (y(k + 1) - \bar{y}(1|k))$:

$$V_{\gamma}(x(k + 1), \tilde{u}(k + 1)) - V_{\gamma}(x^*(1|k), \tilde{u}(k + 1)) \leq \nu(e_i(k + 1)).$$

(20)

Then, following the previous steps, it can be derived that

$$V_{\gamma}(x(k + 1) \leq V_{\gamma}(x(k + 1), \tilde{u}(k + 1)) \leq \nu(e_i(k + 1) + V_{\gamma}^N(x(k)) - \tilde{\xi}(x(k), u(k)).$$

Thus, $V_{\gamma}(x)$ is an ISS Lyapunov function (Limon et al., 2008).

**Remark 8 (Suboptimal Case).** The stability analysis can be extended to the case in which the optimal solution of the control problem is not found. Given an initial feasible solution of the control problem, if the optimizer is able to improve the cost (even for a suboptimal solution of the problem), then the controller is able to maintain robust stability while satisfying the constraints (Rawlings & Mayne, 2009).

**Remark 9 (Violation of Assumption 2).** If the bound on the prediction error is estimated from data (e.g., via cross-validation), then the error could take a value larger than $\mu$ for a certain period of time. In this case, the ISS property (21) still holds as long as $x(k) \in \Gamma$ for that period of time. Notice that the ISS condition is derived from the smoothness of the optimal cost function, which is an inherent property of the proposed optimal control problem.

**Remark 10 (Stability Margin).** Most of the robust controllers for constrained systems exhibit an upper bound on the estimation error to be solvable (Limon et al., 2008). This is the so-called stability margin. One of the main drawbacks of robust predictive controllers is that this margin is typically quite conservative, due to the open-loop nature of the predictions. As a robust controller, our approach inherits this drawback.

### 4. Case study

In this section, the proposed controller is applied to a continuously-stirred tank reactor (Manzano et al., 2019), in which a reaction $A \rightarrow B$ takes place. The system’s state is defined by the concentration of the reactant, $C_A$ (mol/l), the temperature in the tank, $T$ (K), and the temperature of the coolant, $T_c$ (K). The state varies with respect to the control input, which is the reference temperature of the coolant, $T_f$ (K), according to the following set of ordinary differential equations (ODEs), which will be used to simulate the system but are assumed to be unknown:

$$\frac{dC_A}{dt} = \frac{q_0}{V} \cdot (C_M - C_A(t))$$

$$\frac{dT}{dt} = \frac{q_0}{V} \cdot (T_f - T(t))$$

(22a)
The proposed controller is applied in 100 closed-loop simulations, subject to random noise. The results are shown in the last row of Fig. 3. Note that the output is steered to the reference while the constraints are satisfied. The optimization problem is solved in Matlab on an Intel® Core™ i7-6700HQ CPU @ 2.60 GHz 12GB RAM and each iteration takes less than one second to complete, much shorter than the 30 s required by the sampling time.

In order to compare the proposed controller to other methods, the same setup is simulated with two different MPCs. First, a controller derived from (17), but with the set of ODEs (22) as the state-feedback prediction model, for which \( \mu \) is the maximum noise, 0.025 mol/l. This aims to resemble the ideal case of perfect knowledge of the plant (result shown in the first row of Fig. 3).

As expected, the data-based controller performs slower than the constraints are satisfied. The optimization problem is solved in Matlab on an Intel® Core™ i7-6700HQ CPU @ 2.60 GHz 12GB RAM and each iteration takes less than one second to complete, much shorter than the 30 s required by the sampling time.

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References


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