Super-multiplicativity and a lower bound for the decay of the signature of a path of finite length

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Abstract

For a path of length \( L > 0 \), if for all \( n \geq 1 \), we multiply the \( n \)-th term of the signature by \( n!L^{-n} \), we say the resulting signature is ‘normalised’. It has been established[3] that the norm of the \( n \)-th term of the normalised signature of a bounded-variation path is bounded above by 1. In this article we discuss the super-multiplicativity of the norm of the signature of a path with finite length, and prove by Fekete’s lemma the existence of a non-zero limit of the \( n \)-th root of the norm of the \( n \)-th term in the normalised signature as \( n \) approaches infinity.

Résumé

Pour une trajectoire de longueur \( L > 0 \), si l’on multiplie le \( n \)-ième terme de la signature par \( n!L^{-n} \) pour tout \( n \geq 1 \), on la signale ainsi
obtenue est dite "normalisée". Il a été établi en [3] que la norme du n-ième terme de la signature normalisée d’une trajectoire à variation bornée est majorée par 1. Dans cet article nous étudions la super-multiplicativité de la norme de la signature d’une trajectoire de longueur finie, et nous démontrons à l’aide du lemme de Fekete l’existence d’une limite non nulle lorsque n tend l’infini pour la racine n-ième de la norme du n-ième terme de la signature normalisée.

1 Super-multiplicativity of the signature in reasonable tensor algebra norms

Definition 1. Let \( \{V_j\}_{j=1}^N \) be normed vector spaces over \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{C} \). Their algebraic tensor product space is defined as the vector space

\[
V_1 \otimes \ldots \otimes V_N = \left\{ \sum_{i \in I} v_1^i \otimes \ldots \otimes v_N^i : v_j^i \in V_j, \quad \forall i \in I, |I| < \infty, J = 1, \ldots, N. \right\},
\]

where we identify \((u + v) \otimes w = u \otimes w + v \otimes w\).

Definition 2. If \( \phi_j \in V_j^* \) are bounded linear functionals on \( V_j, j = 1, \ldots, N \), then we define the dual action of \( \phi_1 \otimes \ldots \otimes \phi_N \) on \( V_1 \otimes \ldots \otimes V_N \to \mathbb{F} \) by

\[
(\phi_1 \otimes \ldots \otimes \phi_N)(\sum_{i \in I} v_1^i \otimes \ldots \otimes v_N^i) := \sum_{i \in I} \prod_{j=1}^N \phi_j(v_j^i)
\]

for all \( v_j^i \in V_j, j = 1, \ldots, N, i \in I, |I| < \infty \). The map is well-defined and independent of the representation on the right-hand side.

Now we state the properties of the norms on tensor products that are required for this article.

Definition 3 (Reasonable tensor algebra norm). Let \( V, V \otimes V, \ldots, V^\otimes n \) be normed vector spaces. We assume that for all \( v \in V^\otimes n, w \in V^\otimes m \),

\[
\|v \otimes w\| \leq \|v\|\|w\|
\]

and the norm induced on the dual spaces satisfies that for all \( \phi \in (V^\otimes m)', \psi \in (V^\otimes n)' \),

\[
\|\phi \otimes \psi\| \leq \|\phi\|\|\psi\|.
\]

Moreover, if \( S(n) \) denotes the symmetric group over \( \{1, 2, \ldots, n\} \), we assume that for all \( n \geq 1 \),

\[
\|\sigma(v)\| = \|v\| \quad \forall \sigma \in S(n), v \in V^\otimes n.
\]
Proposition 1 (Ryan[4]). Let $X$ and $Y$ be normed vector spaces. If $\| \| \|$ is a tensor norm on $X \otimes Y$ which satisfies

$$\| v \otimes w \| \leq \| v \| \| w \| \quad \forall v \in X, w \in Y;$$

and the norm induced on the dual spaces satisfies

$$\| \phi \otimes \psi \| \leq \| \phi \| \| \psi \| \quad \forall \phi \in X', \psi \in Y',$$

then $\| \| \|$ is called a reasonable cross norm, and $\| x \otimes y \| = \| x \| \| y \|$ for every $x \in X$ and $y \in Y$; for every $\phi \in X'$ and $\psi \in Y'$, the norm of the linear functional $\phi \otimes \psi$ on $(X \otimes Y, \| \| \|)$ satisfies $\| \phi \otimes \psi \| = \| \psi \| \| \phi \|$.

Using Proposition 1 implies that the inequalities in Equation (1) and (2) imply equality.

Remark 1. Note that under the assumptions of Definition 3 for all $a \in V_1 \otimes \ldots \otimes V_N$, $b \in V_m \otimes V_n$, $c \in V_l \otimes V_m$, $\| (a \otimes b) \otimes c \| = \| a \otimes (b \otimes c) \| = \| a \| \| b \| \| c \|.$

We provide some examples of tensor norms which are reasonable tensor algebra norms.

Definition 4. Let $\{V_j\}_{j=1}^N$ be normed vector spaces over $F$. The projective tensor norm on $V_1 \otimes \ldots \otimes V_N$ is defined such that for $x = \sum_{i \in I} v_i^1 \otimes \ldots \otimes v_i^N, v_i^j \in V_j \forall i \in I, |I| < \infty$,

$$\| x \|_\pi := \inf \left\{ \sum_{i \in I} \| v_i^1 \| \ldots \| v_i^N \| : x = \sum_{i \in I} v_i^1 \otimes \ldots \otimes v_i^N, v_i^j \in V_j \forall i \in I, |I| < \infty \right\}.$$

The injective tensor norm on $V_1 \otimes \ldots \otimes V_N$ is defined such that for $x = \sum_{i \in I} v_i^1 \otimes \ldots \otimes v_i^N \in V_1 \otimes \ldots \otimes V_N, i \in I, |I| < \infty$,

$$\| x \|_\delta := \sup \left\{ \| \sum_{i \in I} \prod_{j=1}^N \phi_j(v_i^j) \| : \phi_j \in V_j', \| \phi_j \| \leq 1 \forall j = 1, \ldots, N \right\}$$

for any representation of $x$.

Lemma 1. The projective tensor norm and the injective tensor norm defined in Definition 4 both satisfy the properties stated in Definition 3. Moreover, if $\alpha$ is a reasonable cross norm on $X \otimes Y$, and $u \in X \otimes Y$, then

$$\| x \|_\delta \leq \alpha(x) \leq \| x \|_\pi.$$

Furthermore, any reasonable tensor algebra norm is sandwiched between the injective and projective tensor norms.
The proof of Lemma 1 is omitted here.

**Lemma 2.** The Hilbert-Schmidt norm is a reasonable tensor algebra norm.

The proof of Lemma 2 is omitted here.

**Definition 5.** Let $V, V \otimes V, ..., V \otimes^n$ be Banach completed spaces equipped with a reasonable tensor algebra norm compatible with the norm on $V$, and $\gamma : J \to V$ be a continuous path with finite length. The signature of $\gamma$ is denoted by

$$S = (1, S_1, S_2, ..., S_n, ...)$$

where for each $n \geq 1$, $S_n = \int_{u_1 < \ldots < u_n, a_1, \ldots, a_n \in J} d\gamma_{u_1} \otimes ... \otimes d\gamma_{u_n}$.

**Remark 2.** Note that the $n$-th term of $S$ lives in the completed Banach space $V \otimes^n$ whenever the algebraic tensor product is completed with a reasonable tensor algebra norm.

From now on we will fix a Banach space $V$, a reasonable tensor algebra norm, and we will take $V \otimes^n$ to be the completion of the algebraic tensor product with respect to that reasonable tensor algebra norm.

**Definition 6** (Shuffle product). The shuffle product is defined inductively to be bilinear, and so that

$$u \otimes a \shuffle v \otimes b := (u \shuffle v \otimes b) \otimes a + (u \otimes a \shuffle v) \otimes b$$

for any $a, b \in V$.

**Definition 7** (Group-like elements). Define

$$\tilde{T}(V) := \{ (a_0, a_1, a_2, ...) : a_n \in V \otimes^n, \forall n \geq 1, a_0 = 1 \}.$$

An element $a \in \tilde{T}(V)$ is called group-like if for all $\phi, \psi \in (\tilde{T}(V))^\prime$,

$$\phi \shuffle \psi(a) = \phi(a) \psi(a).$$

**Theorem 1.** Suppose $\gamma : J \to V$ is a path of finite length. Then for $m, n \geq 0$, the signature of $\gamma$ satisfies

$$\| (m + n)!S_{m+n} \| \geq \| n!S_n \| \cdot \| m!S_m \| \quad \forall m, n \geq 0.$$ (4)

where $\| \|$ is any reasonable tensor algebra norm. $V \otimes^0$ is defined to be $\mathbb{F}$, and $S_0 = 1$. 

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Proof. By Hahn-Banach Theorem, there exists $\phi_n \in (V \otimes n)'$, $\phi_m \in (V \otimes m)'$ such that $\|\phi_n\| = 1$, $\|\phi_m\| = 1$, and

$$\phi_n(S_n) = \|S_n\|, \; \phi_m(S_m) = \|S_m\|.$$ 

Equivalently, we can write

$$\phi_n(S) = \|S_n\|, \; \phi_m(S) = \|S_m\|,$$

where we define $\phi_k(x) = 0$ for $x \notin V \otimes k$ for all $k \geq 0$. From [3] we know that $S$ is group-like, hence

$$\phi_m \sqcup \phi_n(S) = \phi_m(S)\phi_n(S) = \|S_m\|\|S_n\|.$$ 

Also,

$$\phi_m \sqcup \phi_n(S_{m+n}) = \sum_{\sigma \in \text{Shuffles}(m,n)} \sigma(\phi_m \otimes \phi_n)(S_{m+n}) = \sum_{\sigma \in \text{Shuffles}(m,n)} (\phi_m \otimes \phi_n)(\sigma^{-1}(S_{m+n})),$$

so

$$|\phi_m \sqcup \phi_n(S_{m+n})| \leq \#\text{shuffles}(m,n)\|\phi_m \otimes \phi_n\|\|S_{m+n}\|.$$ 

Note that $\#\text{shuffles}(m,n) = \frac{(m+n)!}{m!n!}$, and by Definition 3 we know that

$$\|\phi_m \otimes \phi_n\| \leq \|\phi_m\||\phi_n\| = 1.$$ 

Hence

$$\|(m + n)!S_{m+n}\| \geq \|n!S_n\|\|m!S_m\|$$

as expected. \qed

**Corollary 1.** If $S_j = 0$, then $S_k = 0$ for $k = 1, \ldots, j$.

**Proof.** The proof follows from Theorem 1. \qed


2 Limiting behaviour

We note the following lemma by Fekete[5].

**Theorem 2** (Fekete’s Lemma). If a sequence of real numbers \( \{a_n\}_{n \in \mathbb{N}} \) satisfies the sub-additivity condition

\[
a_{m+n} \leq a_m + a_n \quad \forall m, n \in \mathbb{N},
\]

Then

\[
\lim_{n \to \infty} \frac{a_n}{n} = \inf_{n \in \mathbb{N}} \frac{a_n}{n}.
\]

**Theorem 3** (Asymptotic behaviour of the signature). If \( \gamma : J \to V \) is a continuous tree-reduced path of finite length \( L > 0 \), then under any reasonable tensor algebra norm \( \| \| \), there exists a non-zero limit \( \tilde{L} \) such that

\[
\lim_{n \to \infty} \frac{\| n!S_n \|^{1/n}}{\| m!S_m \|^{1/n}} = \sup_{k \geq 1} \frac{\| k!S_k \|^{1/k}}{\| (m+n)!S_{m+n} \|^{1/k}} = \tilde{L} > 0.
\]

**Proof.** By Theorem 1, we know that for all \( m, n \geq 0 \),

\[
\| (m+n)!S_{m+n} \| \geq \| n!S_n \| \| m!S_m \|.
\]

Taking logarithm gives

\[
-\log(\| (m+n)!S_{m+n} \|) \leq -\log(\| n!S_n \|) - \log(\| m!S_m \|).
\]

So the function \( f(n) := -\log(\| n!S_n \|/L^n) \) satisfies \( f(m+n) \leq f(m) + f(n) \) for all \( m, n \in \mathbb{N} \). Then by Fekete’s lemma[5], \( \frac{1}{n} \log(\| n!S_n \|) \) converges to \( \sup_{k \geq 1} \log(\| k!S_k \|)/k \); hence \( \| n!S_n \|^{1/n} \) converges to \( \sup_{k \in \mathbb{N}} \| k!S_k \|^{1/k} \). Note by Hambly and Lyons[2], every path of finite length has a unique tree-reduced version with the same signature, if the tree-reduced path is non-trivial then there will be at least one term in the signature of the path which is non-zero. Hence \( \sup_{k \geq 1} \| k!S_k \|^{1/k} \) is non-zero. Therefore \( \| n!S_n \|^{1/n} \) converges to a non-zero limit as \( n \) increases. 

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1. Roughly speaking, a tree-reduced path is the a path where it does not go back on cancelling itself over any interval.
Corollary 2. Let $V$ be a Banach space. For any element 
\[ a = (a_0, a_1, a_2, \ldots) \in \{ (b_0, b_1, b_2, \ldots) : b_0 = 1, b_n \in V^\otimes n \forall n \geq 1 \} \]
which is group-like, we have 
\[ \| (m + n)!a_{m+n} \| \geq \| m!a_m \| \| n!a_n \| \quad \forall m, n \geq 0, \]
and \( \| n!a_n \|^{1/n} \) converges to \( \sup_{k \in \mathbb{N}} \| k!a_k \|^{1/k} \) as \( n \) increases under any reasonable tensor algebra norm \( \|. \| \).

Proof. Note that since \( a \) is group-like, the same arguments apply as in Theorem 1 and Theorem 3.

Remark 3. It is an interesting question to ask whether there is an nice and simple form of the limit of \( \| n!S_n \|^{1/n} \) mentioned in Theorem 3, and whether the limit is the same under any reasonable tensor algebra norm. Moreover, we know from [3] that for a path with finite length \( L > 0 \), an upper bound of \( \| n!S_n \| \) is \( L^n \). Furthermore, Lyons and Hambly[2] proved that for a smooth enough path of finite length, the ratio \( \| n!S_n \| / L^n \) converges to 1 under certain norms. Therefore we have the following conjecture.

Conjecture 1. Let \( V \) be a Banach space, and \( \gamma : J \rightarrow V \) be a path with finite length \( L > 0 \). Then the signature of \( \gamma \) satisfies that 
\[ \| n!S_n \|^{1/n} \rightarrow L \quad \text{as} \quad n \rightarrow \infty, \]
under any reasonable tensor algebra norm.

Remark 4. An interesting tensor norm to consider is the Haagerup tensor norm[1]. Clearly the Haagerup norm is not a reasonable tensor algebra norm, however under the Haagerup norm, for a path of finite length \( L > 0 \), we still have \( n!\| S_n \| \leq L^n \). Therefore it is an interesting question to ask whether the signature will have the same behaviour as described in Theorem 3 under the Haagerup tensor norm, or the symmetrised forms of the Haagerup tensor norm.

Remark 5. Although it has been shown that \( \| n!S_n \| \) eventually behaves like \( L^n \) under certain norms for well-behaved paths(see [2]), some simple examples show that in general for a path with finite length, \( \| n!S_n \| / L^n \) does not necessarily converge to 1 as \( n \) increases. Therefore the result in Theorem 3 is the best description we can have about the decay of the signature for a path with finite length.

For a \( p \)-variation path where \( p > 1 \), by considering simple examples we can see that we cannot have a non-zero limit for \( \|(n/p)!S_n\|^{1/n} \) as \( n \) increases.
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References


