

An Iterated Azéma-Yor Type Embedding for Finitely Many Marginals

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Abstract

We solve the n -marginal Skorokhod embedding problem for a continuous local martingale and a sequence of probability measures μ_1, \dots, μ_n which are in convex order and satisfy an additional technical assumption. Our construction is explicit and is a multiple marginal generalisation of the Azéma and Yor [1] solution. In particular, we recover the stopping boundaries obtained by Brown et al. [4] and Madan and Yor [14]. Our technical assumption is necessary for the explicit embedding, as demonstrated with a counterexample. We discuss extensions to the general case giving details when $n = 3$.

In our analysis we compute the law of the maximum at each of the n stopping times. This is used in Henry-Labordère et al. [10] to show that the construction maximises the distribution of the maximum among all solutions to the n -marginal Skorokhod embedding problem. The result has direct implications for robust pricing and hedging of Lookback options.

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1 Introduction

We consider here an n -marginal Skorokhod embedding problem (SEP). We construct an explicit solution which has desirable optimal properties. The classical (one-marginal) SEP consists in finding a stopping time τ such that a given stochastic process (X_t) stopped at τ has a given distribution μ . For the solution to be useful (and non-trivial) one further requires τ to be *minimal* (cf. Obłój [17, Sec. 8]). When X is a continuous local martingale and μ is centred in X_0 , this is equivalent to $(X_{t \wedge \tau} : t \geq 0)$ being a uniformly integrable martingale. The problem, dating back to the original work in Skorokhod [20], has been an active field of research for nearly 50 years. New solutions often either considered new classes of processes X or focused on finding stopping times τ with additional optimal properties. This paper contributes to the latter category. We are motivated, as was the case for several earlier works in the field, by questions arising in mathematical finance which we highlight below.

The problem and main results. To describe the problem we consider, take a standard Brownian motion B and a sequence of probability measures μ_1, \dots, μ_n . A solution to the n -marginal SEP is a sequence of stopping times $\tau_1 \leq \dots \leq \tau_n$ such that $B_{\tau_i} \sim \mu_i$, $1 \leq i \leq n$, and $(B_{t \wedge \tau_n})_{t \geq 0}$ is a uniformly integrable martingale. It follows from Jensen's inequality that a solution may exist only if all μ_i are centred and the sequence is in convex order. And then it is easy to see how to solve the problem: it suffices to iterate a solution to the classical case $n = 1$ developed for a non-trivial initial distribution of B_0 , of which several exist.

In contrast, the question of optimality is much more involved. In general there is no guarantee that a simple iteration of optimal embeddings would be globally optimal. Indeed, this is usually not the case. Consider the embedding of Azéma and Yor [1] which consists of a first exit time for the joint process $(B_t, \bar{B}_t)_{t \geq 0}$, where $\bar{B}_t = \sup_{s \leq t} B_s$. More precisely, their solution $\tau^{\text{AY}} = \inf \{t \geq 0 : B_t \leq \xi_\mu(\bar{B}_t)\}$ leads to a functional relation $B_{\tau^{\text{AY}}} = \xi_\mu(\bar{B}_{\tau^{\text{AY}}})$. This then translates into the optimal property that the distribution of $\bar{B}_{\tau^{\text{AY}}}$ is maximized in stochastic order amongst all solutions to SEP for μ , i.e. for all y ,

$$\mathbb{P}[\bar{B}_{\tau^{\text{AY}}} \geq y] = \sup \{ \mathbb{P}[\bar{B}_\rho \geq y] : \rho \text{ s.t. } B_\rho \sim \mu, (B_{t \wedge \rho}) \text{ is UI} \}.$$

It is not hard to generalise the Azéma-Yor embedding to a non-trivial starting law, see Obłój [17, Sec. 5]. Consequently we can find η_i such that $\tau_i = \inf \{t \geq \tau_{i-1} : B_t \leq \eta_i(\sup_{\tau_{i-1} \leq s \leq t} B_s)\}$ solve the n -marginal SEP. However this construction will maximise stochastically the distributions of $\sup_{\tau_{i-1} \leq t \leq \tau_i} B_t$, for each $1 \leq i \leq n$, but not of the global maximum \bar{B}_{τ_n} . The latter is achieved with a new solution which we develop here.

Our construction involves an interplay between all n -marginals and hence is not an iteration of a one-marginal solution. However it preserves the spirit of the Azéma-Yor embedding in the following sense. Each τ_i is still a first exit for $(B_t, \bar{B}_t)_{t \geq \tau_{i-1}}$ which is designed in such a way as to obtain a “strong relation” between B_{τ_i} and \bar{B}_{τ_i} , ideally a functional relation. Under our technical assumption about the measures μ_1, \dots, μ_n , Assumption \otimes , we describe this relation in detail in Lemma 3.1.

For $n = 2$ we recover the results of Brown et al. [3]. We also recover the trivial case $\tau_i = \tau_{\mu_i}^{\text{AY}}$ which happens when $\xi_{\mu_i} \leq \xi_{\mu_{i+1}}$, we refer to Madan and Yor [14] who in particular then investigate properties of the arising time-changed process. However, as a counterexample shows, our construction does not work for all laws μ_1, \dots, μ_n which are in convex order. Assumption \otimes fails when a special interdependence between the marginals is present and the analysis then becomes more technical and the resulting quantities are, in a way, less explicit. We only detail the appropriate arguments for the case $n = 3$.

We stress that the problem considered in this paper is significantly more complex than the special case $n = 1$. For $n = 1$ several solutions to SEP exist with different optimal properties. For $n = 2$ only one such construction, the generalisation of the Azéma–Yor embedding obtained by Brown et al. [4], seems to be known. To the best of our knowledge, the solution we present here is the first one to deal with the general n -marginal SEP.

Motivation and applications. Our results have direct implications for, and were motivated by, robust pricing and hedging of lookback options. In mathematical finance, one models the price process S as a martingale and specifying prices of call options at maturity T is equivalent to fixing the distribution μ of S_T . Understanding no-arbitrage price bounds for a functional O , which time-changes appropriately, is then equivalent to finding the range of $\mathbb{E}[O(B)_\tau]$ among all solutions to the Skorokhod embedding problem for μ . This link between SEP and robust pricing and hedging was pioneered by Hobson [11] who considered Lookback options. Barrier options were subsequently dealt with by Brown et al. [3]. More recently, Cox and Oblój [6, 7] considered the case of double touch/no-touch barrier options, Hobson and Neuberger [13] looked at forward starting straddles and analysis for variance options was undertaken by Cox and Wang [8]. We refer to Hobson [12] and Oblój [18] for an exposition of the main ideas and more references. However, all the previous works considered essentially the case of call options with one maturity, i.e. a one-marginal SEP, while in practice prices for many intermediate maturities may also be available. This motivated our investigation.

We started our quest for a general n -marginal optimal embedding by computing the value function $\sup \mathbb{E}[\phi(\sup_{t \leq \tau_n} B_t)]$ among all solutions to the n -marginal SEP. This was achieved using stochastic control methods, developed first for $n = 1$ by Galichon et al. [9], and is reported in a companion paper by Henry-Labordère et al. [10]. Knowing the value function we could start guessing the form of the optimiser and this led to the present paper. Consequently the optimal properties of our embedding, namely that it indeed achieves the value function in question, are shown by Henry-Labordère et al. [10]. In fact we give two proofs in that paper, one via stochastic control methods and another one by constructing appropriate pathwise inequalities and exploiting the key Lemma 3.1 below, cf. Henry-Labordère et al. [10, Section 4].

Organisation of the paper. The remainder of the paper is organized as follows. In Section 2 we explain the main quantities for the embedding and state the main result. We also present the restriction on the measures μ_1, \dots, μ_n which we require for our construction to work (Assumption \circledast). In Section 3 we prove the main result and Section 4 provides a discussion of extensions together with comments on Assumption \circledast . The proof of an important but technical lemma is relegated to the Appendix.

2 Main Result

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)$, be a filtered probability space satisfying the usual hypothesis and B a continuous \mathbb{F} -local martingale, $B_0 = 0$, $\langle B \rangle_\infty = \infty$ a.s. and B has no intervals of constancy a.s. We denote $\bar{B}_t := \sup_{s \leq t} B_s$. We are primarily interested in the case when B is a standard Brownian motion and it is convenient to keep this example in mind, hence the notation. We allow for more generality as this introduces no changes to the statements or the proofs.

2.1 Definitions

The following definition will be crucial in the remainder of the article. We define the stopping boundaries ξ_1, \dots, ξ_n for our iterated Azéma-Yor type embedding together with quantities K_1, \dots, K_n which will be later linked to the law of the maximum at subsequent stopping times.

Definition 2.1. Fix $n \in \mathbb{N}$. For convenience we set

$$c_0 \equiv 0, \quad K_0 \equiv 0, \quad \xi_0 \equiv -\infty. \quad (2.1)$$

For $\zeta \in \mathbb{R}$ and $i = 1, \dots, n$ we write

$$c_i(\zeta) := \int_{\mathbb{R}} (x - \zeta)^+ \mu_i(dx). \quad (2.2)$$

Let $y \geq 0$ and assume that for $i = 1, \dots, n-1$ the quantities ξ_i, K_i, ν_i and J_i are already defined. Then we define

$$\begin{aligned} \nu_n(\cdot; y) &: (-\infty, y] \rightarrow \{0, 1, \dots, n-1\}, \\ \zeta \mapsto \nu_n(\zeta; y) &:= \max \{k \in \{0, 1, \dots, n-1\} : \xi_k(y) < \zeta\}, \end{aligned} \quad (2.3)$$

and

$$\xi_n(y) := \sup \left\{ \arg \inf_{\zeta < y} \left(\frac{c_n(\zeta)}{y - \zeta} - \left[\frac{c_{\nu_n(\zeta; y)}(\zeta)}{y - \zeta} - K_{\nu_n(\zeta; y)}(y) \right] \right) \right\}. \quad (2.4)$$

With

$$J_n(y) := \nu_n(\xi_n(y); y) \quad (2.5)$$

we set

$$K_n(y) := \frac{1}{y - \xi_n(y)} \left\{ c_n(\xi_n(y)) - [c_{J_n(y)}(\xi_n(y)) - (y - \xi_n(y))K_{J_n(y)}(y)] \right\}. \quad (2.6)$$

Definition 2.2 (Embedding). Set $\tau_0 \equiv 0$ and for $i = 1, \dots, n$ define

$$\tau_i := \begin{cases} \inf \{t \geq \tau_{i-1} : B_t \leq \xi_i(\bar{B}_t)\} & \text{if } B_{\tau_{i-1}} > \xi_i(\bar{B}_{\tau_{i-1}}), \\ \tau_{i-1} & \text{else.} \end{cases} \quad (2.7a)$$

$$(2.7b)$$

Remark 2.3 (Properties of ν_n). Recalling the definition of ν_n , cf. (2.3), we observe for later use that for $y \geq 0$

$$\nu_n(\cdot; y) \text{ is left-continuous and has at most } n-1 \text{ jumps} \quad (2.8)$$

and for $x \in \mathbb{R}$

$$\nu_n(x; \cdot) \text{ is right-continuous and has at most } n-1 \text{ jumps.} \quad (2.9)$$

Figure 2.1 illustrates a set of possible stopping boundaries ξ_1, ξ_2, ξ_3 in the case of $n = 3$. If Assumption \circledast is in place, see Section 2.2, we will show that the stopping boundaries are continuous (except possibly for $i = 1$) and non-decreasing, cf. Section 2.5.

The n^{th} stopping boundary ξ_n is obtained from an optimization problem which features ξ_1, \dots, ξ_{n-1} and K_1, \dots, K_{n-1} . $K_n(y)$ is the value of the objective function at the optimal value $\xi_n(y)$. Note that all previously defined stopping boundaries ξ_1, \dots, ξ_{n-1} and the quantities K_1, \dots, K_{n-1} remain unchanged.

the index of the last law $\mu_i, i < n$, which represents, locally at level of maximum y , a *binding constraint* for the embedding. As compared to the optimization from which b_n^{-1} is obtained, cf. (2.12), the optimization from which ξ_n is obtained, cf. (2.4), has a penalty term.

2.2 Restrictions on Measures

Throughout the article we will denote the left- and right-limit of a function f at x (if it exists) by $f(x-)$ and $f(x+)$, respectively.

Recalling the conventions in (2.1), we define inductively for $n \in \mathbb{N}$ and $y \geq 0$ the mappings

$$\begin{aligned} c^n(\cdot, y) &: (-\infty, y] \rightarrow \mathbb{R} \cup \{\infty\}, \\ x \mapsto c^n(x, y) &:= c_n(x) - [c_{\iota_n(x;y)}(x) - (y-x)K_{\iota_n(x;y)}(y)]. \end{aligned} \quad (2.13)$$

It follows that the minimization problem in (2.4) is equivalent to the following minimization problem,

$$\xi_n(y) \in \arg \min_{\zeta \leq y} \frac{c^n(\zeta, y)}{y - \zeta}, \quad (2.14)$$

where we observe that

$$\begin{aligned} \frac{c^n(\zeta, y)}{y - \zeta} \Big|_{\zeta=y} &:= \lim_{\zeta \uparrow y} \frac{c^n(\zeta, y)}{y - \zeta} \\ &= \begin{cases} -c'_n(y-) + c'_{\iota_n(y;y)}(y-) + K_{\iota_n(y;y)}(y) & \text{if } c_n(y) = c_{\iota_n(y;y)}(y), \\ +\infty & \text{else.} \end{cases} \end{aligned} \quad (2.15)$$

Now we want to argue existence in (2.14). In the case $y > 0$ this can be deduced iteratively from the a priori piecewise continuity of $c^n(\cdot, y)$ and the fact that $c^n \geq 0$ together with the property that $\zeta \mapsto \frac{c^n(\zeta, y)}{y - \zeta} = \frac{c_n(\zeta)}{y - \zeta}$ for ζ sufficiently small, which is a non-increasing function. This is because, inductively, $\xi_1(y), \dots, \xi_{n-1}(y)$ are finite and fixed and hence $\iota_n(\zeta; y) = 0$ for $\zeta < \min_{i < n} \xi_i(y)$.

For $y \geq 0$, we extend

$$\frac{c^n(\zeta, y)}{y - \zeta} \Big|_{\zeta=l_{\mu_n}} := \begin{cases} \frac{-l_{\mu_n}}{y-l_{\mu_n}}, & \text{if } l_{\mu_n} > -\infty, \\ 1 & \text{else.} \end{cases} \quad (2.16)$$

For later use observe

$$\min_{i \leq n} b_i^{-1}(y) \leq \xi_n(y) \leq y \quad (2.17)$$

which follows from the definition of ξ_n , cf. (2.4), and where b_i^{-1} denotes the right-continuous inverse of the barycentre function b_i , cf. (2.12).

Assumption \otimes (Restriction on Measures). *Recall definitions in (2.1)–(2.2), (2.13) and (2.10). We impose the following restrictions on the measures μ_1, \dots, μ_n :*

- (i) $\int |x| \mu_i(dx) < \infty$ with $\int x \mu_i(dx) = 0$ and $c_{i-1} \leq c_i$ for all $1 \leq i \leq n$,
- (ii) for all $2 \leq i \leq n$ and all $0 < y < r_{\mu_i}$ the mapping

$$[l_{\mu_i}, y] \rightarrow \mathbb{R} \cup \{+\infty\}, \quad \zeta \mapsto \frac{c^i(\zeta, y)}{y - \zeta} \quad \text{has a unique minimizer } \zeta^* \quad (2.18)$$

and

$$c_i(y) > c_{\iota_i(y;y)}(y) \quad \text{whenever } \zeta^* < y. \quad (2.19)$$

Remark 2.4 (Assumption \otimes). The condition that the call prices are non-decreasing in maturity

$$c_i \leq c_{i+1}, \quad i = 1, \dots, n-1, \quad (2.20)$$

can be rephrased by saying that μ_1, \dots, μ_n are non-decreasing in the convex order. Condition (i) in Assumption \otimes is the necessary and sufficient condition for a uniformly integrable martingale with these marginals to exist, as shown by e.g. Strassen [21, Theorem 2] or Meyer [15, Chapter XI].

Condition (ii) in Assumption \otimes will be discussed further in section 4.

Note that if (2.20) holds with strict inequality then (2.19) is automatically satisfied.

Remark 2.5 (Discontinuity of ξ_1). Note that Assumption \otimes (ii) does not require that the mapping

$$\zeta \mapsto \frac{c^1(\zeta, y)}{y - \zeta} = \frac{c_1(\zeta, y)}{y - \zeta} \quad (2.21)$$

has a unique minimizer. It may happen that there is an interval of minimizers and then ξ_1 is discontinuous at such y .

2.3 The Main Result

Our main result shows how to iteratively define an embedding of (μ_1, \dots, μ_n) in the spirit of Azéma and Yor [1] and Brown et al. [4] if Assumption \otimes is in place.

Theorem 2.6 (Main Result). *Let $n \in \mathbb{N}$ and assume that the measures μ_1, \dots, μ_n satisfy Assumption \otimes from Section 2.2. Recall Definitions 2.1 and 2.2. Then $\tau_i < \infty$, $B_{\tau_i} \sim \mu_i$ for all $i = 1, \dots, n$ and $(B_{\tau_n \wedge t})_{t \geq 0}$ is a uniformly integrable martingale.*

In addition, we have for $y \geq 0$ and $i = 1, \dots, n$,

$$\mathbb{P}[\bar{B}_{\tau_i} \geq y] = K_i(y) \quad (2.22)$$

where K_i is defined in (2.6).

Remark 2.7 (Inductive Nature). It is important to observe that ξ_i and therefore also τ_i , only depend on μ_1, \dots, μ_i . This gives an iterative structure allowing to “add one marginal at a time” and enables us to naturally prove the Theorem by induction on n .

Remark 2.8 (Minimality). Since all τ_i are such that $(B_{t \wedge \tau_i})_{t \geq 0}$ is a uniformly integrable martingale it follows from Monroe [16] that all τ_i are *minimal*.

2.4 Examples

Examples 2.9 and 2.10, respectively, show that we recover the stopping boundaries obtained by Madan and Yor [14] and Brown et al. [4], respectively. In particular the case $n = 1$ corresponds to the solution of Azéma and Yor [1].

Example 2.9 (Madan and Yor [14]). *Recall the definition of the barycentre function b_i from (2.11). Madan and Yor [14] consider the “increasing mean residual value” case, i.e.*

$$b_1 \leq b_2 \leq \dots \leq b_n. \quad (2.23)$$

We will now show that our main result reproduces their result if Assumption \otimes is in place. In fact, as can be seen below, our definitions of ξ_i and K_i , cf. (2.4) and

(2.6), respectively, reproduce the correct stopping boundaries in the general case, showing that Assumption \otimes is not necessary, cf. also Section 4. More precisely, we have

$$\xi_i = b_i^{-1}, \quad K_i(y) = \frac{c_i(b_i^{-1}(y))}{y - b_i^{-1}(y)} =: \mu_i^{\text{HL}}([y, \infty)), \quad i = 1, \dots, n, \quad (2.24)$$

where b_i^{-1} denotes the right-continuous inverse of b_i and μ_i^{HL} is the Hardy-Littlewood transform of μ_i , cf. Carraro et al. [5].

Clearly, the claim is true for $i = 1$. Let us assume that the claim holds for all $i \leq n - 1$. Now, the optimization problem for ξ_n in (2.4) becomes

$$\begin{aligned} \xi_n(y) &\in \arg \min_{\zeta \leq y} \left\{ \frac{c_n(\zeta)}{y - \zeta} - \mathbb{1}_{\{\zeta > b_{n-1}^{-1}(y)\}} \left[\frac{c_{n-1}(\zeta)}{y - \zeta} - \frac{c_{n-1}(b_{n-1}^{-1}(y))}{y - b_{n-1}^{-1}(y)} \right] \right\} \\ &\in \arg \min_{\zeta \leq y} \left\{ \min_{\zeta \leq b_{n-1}^{-1}(y)} \frac{c_n(\zeta)}{y - \zeta}, \min_{\zeta \geq b_{n-1}^{-1}(y)} \left(\frac{c_n(\zeta)}{y - \zeta} - \left[\frac{c_{n-1}(\zeta)}{y - \zeta} - \frac{c_{n-1}(b_{n-1}^{-1}(y))}{y - b_{n-1}^{-1}(y)} \right] \right) \right\}. \end{aligned}$$

It is clear that the first minimum is $A_1 = \frac{c_n(b_n^{-1}(y))}{y - b_n^{-1}(y)}$ since $b_n^{-1}(y) \leq b_{n-1}^{-1}(y)$.

As for the second minimum, we set

$$F(\zeta) := \frac{c_n(\zeta)}{y - \zeta} - \left[\frac{c_{n-1}(\zeta)}{y - \zeta} - \frac{c_{n-1}(b_{n-1}^{-1}(y))}{y - b_{n-1}^{-1}(y)} \right]$$

and we see by direct calculation that for almost all $\zeta \in \mathbb{R}$

$$\begin{aligned} (y - \zeta)^2 F'(\zeta) &= (b_n(\zeta) - y) \mu_n([\zeta, \infty)) - (b_{n-1}(\zeta) - y) \mu_{n-1}([\zeta, \infty)) \\ &= c_n(\zeta) \frac{b_n(\zeta) - y}{b_n(\zeta) - \zeta} - c_{n-1}(\zeta) \frac{b_{n-1}(\zeta) - y}{b_{n-1}(\zeta) - \zeta}. \end{aligned}$$

By (2.23), we conclude therefore

$$(y - \zeta)^2 F'(\zeta) \geq (c_n(\zeta) - c_{n-1}(\zeta)) \frac{b_{n-1}(\zeta) - y}{b_{n-1}(\zeta) - \zeta} \geq 0,$$

where the last inequality follows from the non-decrease of the μ_i 's in the convex order. Hence F is non-decreasing, and it follows that it attains its minimum at the left boundary, i.e. $A_2 = \frac{c_n(b_{n-1}^{-1}(y))}{y - b_{n-1}^{-1}(y)} - \left[\frac{c_{n-1}(b_{n-1}^{-1}(y))}{y - b_{n-1}^{-1}(y)} - \frac{c_{n-1}(b_{n-1}^{-1}(y))}{y - b_{n-1}^{-1}(y)} \right] = \frac{c_n(b_{n-1}^{-1}(y))}{y - b_{n-1}^{-1}(y)}$.

Consequently, by (2.12), $\min\{A_1, A_2\} = A_1$ and (2.24) follows.

Example 2.10 (Brown et al. [4]). In the case of $n = 2$ our definition of ξ_1 and ξ_2 clearly recovers the stopping boundaries in the main result of Brown et al. [4]. However, our embedding is not as general as their embedding because we enforce Assumption \otimes , see also the discussion in Section 4.

Example 2.11 (Locally no Constraints). In general we have

$$K_n(y) \leq \mu_n^{\text{HL}}([y, \infty)). \quad (2.25)$$

However, if

$$\xi_n(y) = b_n^{-1}(y) \quad (2.26)$$

for some $y \geq 0$ then it follows from Theorem 2.6 that

$$K_n(y) = \frac{c_n(b_n^{-1}(y))}{y - b_n^{-1}(y)} = \mu_n^{\text{HL}}([y, \infty)), \quad (2.27)$$

i.e. locally at level of maximum y the intermediate laws have no impact on the distribution of the terminal maximum as compared with the (one marginal) Azéma-Yor embedding.

2.5 Properties of ξ_n and K_n

Under Assumption \circledast we establish the continuity of ξ_n for $n \geq 2$, cf. Lemma 2.12, and prove monotonicity of ξ_n for $n \geq 1$, cf. Lemma 2.13. In Lemma 2.14 we derive an ODE for K_n which will be later used to identify the distribution of the maximum of the embedding from Definition 2.2.

Let $n_1 < n_2$. Recalling Remark 2.7 it follows that the embedding of the first n_1 marginals in the n_2 -marginals embedding problem coincides with the n_1 -marginals embedding problem. Hence it is natural to prove the Lemma by induction over the number of marginals n .

Lemma 2.12 (Continuity of ξ_n). *Let $n \geq 2$ and let Assumption \circledast hold. Set*

$$\Delta := \{(x, y) \in \mathbb{R} \times \mathbb{R}_+ : x < y\}. \quad (2.28)$$

Then the mappings

$$c^n : \Delta \rightarrow \mathbb{R}, \quad (x, y) \mapsto c^n(x, y), \quad (2.29)$$

$$\xi_n : \mathbb{R}_+ \rightarrow \mathbb{R}, \quad y \mapsto \xi_n(y) \quad (2.30)$$

are continuous.

Proof. We prove the claim by induction over n . Let us start with the induction basis $n = 1, 2$. Continuity of c^1 is the same as continuity of c_1 and continuity of c^2 is proven by Brown et al. [4], cf. Lemma 3.5 therein. As for continuity of ξ_2 we note that our Assumption \circledast (ii) precisely rules out discontinuities of ξ_2 as shown by Brown et al. [4, Section 3.5]. By induction hypothesis we assume continuity of c^1, \dots, c^{n-1} and ξ_2, \dots, ξ_{n-1} .

The only possibility that a discontinuity of c^n can occur is when the index ι_n changes. This only happens at $(x, y) = (\xi_k(y), y)$ for some $k < n$, or, in the case that y is a discontinuity of ξ_1 , at (x, y) where $x \in [\xi_1(y-), \xi_1(y+)]$. We prove continuity at (x, y) .

Consider first the following cases:

$$\text{if } x = \xi_k(y) \quad \text{then } x \neq \xi_j(y) \text{ for all } j \neq k, j < n, \quad (2.31)$$

$$\text{or, if } x \in [\xi_1(y-), \xi_1(y+)] \quad \text{then } x \neq \xi_j(y) \text{ for all } j \neq 1, j < n. \quad (2.32)$$

Note that in case (2.32) we have from Remark 2.5

$$K_1(y) = \frac{c_1(x)}{y - x} \quad \text{for all } x \in [\xi_1(y-), \xi_1(y+)]. \quad (2.33)$$

We will call a point (x, y) to be “to the right of ξ_k ” if $\xi_k(y) < x$ and “to the left of ξ_k ” if $\xi_k(y) \geq x$. From (2.31) and (2.32) it follows that there exists an $\epsilon > 0$ such that each point (\tilde{x}, \tilde{y}) in the ϵ -neighbourhood of (x, y) is either to the left or to the right of ξ_k and there are no other boundaries in this ϵ -neighbourhood, in particular

$$k = \iota_n(x_r; y_r), \quad j = \iota_n(x_l; y_l) = \iota_{\iota_n(x_r; y_r)}(x_r; y_r), \quad (2.34)$$

where (x_r, y_r) is in the ϵ -neighbourhood of (x, y) and to the right of ξ_k and (x_l, y_l) is in the ϵ -neighbourhood of (x, y) and to the left of ξ_k .

If $x < y$, we have by induction hypothesis

$$c^n(x_r, y_r) = c_n(x_r) - \{c_k(x_r) - (y_r - x_r)K_k(y_r)\} \quad (2.35)$$

$$\begin{aligned} & \xrightarrow[\text{from the right}]{(x_r, y_r) \rightarrow (x, y)} c_n(x) - \{c_k(x) - (y - x)K_k(y)\} \\ & \stackrel{(2.6)}{=} c_n(x) - \left\{ c_k(x) - \frac{y-x}{y-x} \left(c_k(x) - [c_j(x) - (y-x)K_j(y)] \right) \right\} \\ & \stackrel{(2.31)-(2.34)}{=} c_n(x) - [c_j(x) - (y-x)K_j(y)] \\ & \stackrel{(2.13)}{=} c^n(x, y) \quad (2.36) \\ & = c_n(x) - [c_j(x) - (y-x)K_j(y)] \end{aligned}$$

$$\xleftarrow[\text{from the left}]{(x_l, y_l) \rightarrow (x, y)} c_n(x_l) - \{c_j(x_l) - (y_l - x_l)K_j(y_l)\} = c^n(x_l, y_l). \quad (2.37)$$

From (2.35), (2.36) and (2.37) continuity of c^n follows for any sequence $(x_n, y_n) \rightarrow (x, y)$. We now extend the above argument to the situation when $x = y$ which establishes left-continuity of c^n at (y, y) . In this case we have $x = \xi_k(y) = y$. For this to hold we must have $c_k(y) = c_j(y)$. Using boundedness of K_i for $i < n$ shows that (2.36) and (2.37) converge to each other.

To relax (2.31) and (2.32) we successively write out K_k, K_j, \dots , until the assumption of the first case holds true and then, successively, apply the special case.

It remains to prove continuity of ξ_n which we prove by contradiction. Assume there exist $\epsilon > 0$ and $y \geq 0$ such that for all $\delta > 0$ there exists a $y' \in (y, y + \delta)$ such that $|\xi_n(y) - \xi_n(y')| > \epsilon$. By (2.17) the limit of $\xi_n(y')$ as $y' \downarrow y$ exists at least along some subsequence and we denote it by $\tilde{\xi}_n$. By assumption $\tilde{\xi}_n \neq \xi_n(y)$.

Consider first the case that $\xi_n(y) < y$ and $\tilde{\xi}_n < y$. Using continuity of c^n we deduce $\frac{c^n(\xi_n(y'), y')}{y' - \xi_n(y')} \rightarrow \frac{c^n(\tilde{\xi}_n, y)}{y - \tilde{\xi}_n}$ as $y' \rightarrow y$.

Now, if

$$\frac{c^n(\tilde{\xi}_n, y)}{y - \tilde{\xi}_n} \neq \frac{c^n(\xi_n(y), y)}{y - \xi_n(y)} \quad (2.38)$$

then we obtain a contradiction to the optimality of either $\xi_n(y)$ or some $\xi_n(y')$ for y' close enough to y by continuity of c^n . If

$$\frac{c^n(\tilde{\xi}_n, y)}{y - \tilde{\xi}_n} = \frac{c^n(\xi_n(y), y)}{y - \xi_n(y)} \quad (2.39)$$

we obtain a contradiction to Assumption \otimes (ii).

We now consider the case that either $\xi_n(y) = y$ or $\tilde{\xi}_n = y$. The case $\xi_n(y) < y$ and $\tilde{\xi}_n = y$ is ruled out by condition (2.19) from Assumption \otimes (ii): Indeed, for the sequence $\left(K_n(y') = \frac{c^n(\xi_n(y'), y')}{y' - \xi_n(y')} \right)$ to be bounded we must have $c^n(\xi_n(y'), y') \rightarrow 0$. Recalling the left-continuity of c^n at (y, y) implies $c_n(y) = c_{i_n(y; y)}(y)$.

The case $\xi_n(y) = y$ and $\tilde{\xi}_n < y$ follows as above by distinguishing the cases (2.38) and (2.39) and by recalling (2.15) and the left-continuity of c^n at (y, y) . \square

Lemma 2.13 (Monotonicity of ξ_n). *Let $n \in \mathbb{N}$ and let Assumption \otimes hold. Then*

$$\xi_n : \mathbb{R}_+ \rightarrow \mathbb{R}, \quad y \mapsto \xi_n(y) \quad \text{is non-decreasing.} \quad (2.40)$$

Proof. The claim for $n = 1, 2$ follows from Brown et al. [4]. Assume by induction hypothesis that we have proven monotonicity of ξ_1, \dots, ξ_{n-1} .

We follow closely the arguments of Brown et al. [4, Lemma 3.2]. Since ξ_n is continuous it is enough to prove monotonicity at almost every $y \geq 0$. The set of y 's

which are a discontinuity of ξ_1 is a null-set, and hence we can exclude all such y 's. In the following we fix a y where ξ_1, \dots, ξ_n are continuous.

We will first consider the case when $\xi_n(y) \neq \xi_j(y)$ for all $j < n$. By continuity of ξ_n it follows that there is an $\epsilon > 0$ such that

$$\xi_n(\tilde{y}) \neq \xi_j(\tilde{y}) \text{ and } \ell := j_n(y) = j_n(\tilde{y}) \text{ for all } \tilde{y} \in (y - \epsilon, y + \epsilon) \text{ and } j < n, \quad (2.41)$$

and furthermore

$$(\xi_n(\tilde{y}), \tilde{y}) \in (\xi_n(y) - \epsilon, \xi_n(y) + \epsilon) \times (y - \epsilon, y + \epsilon). \quad (2.42)$$

Let l_1 denote a supporting tangent to $c^n(\cdot, y)$ at $\xi_n(y)$ which goes through the x -axis at y , i.e.

$$l_1(x) = c^n(\xi_n(y), y) + (x - \xi_n(y))(D - K_\ell(y)),$$

where D lies between the left- and right-derivatives of $c_n - c_\ell$ at $\xi_n(y)$. Using that $l_1(y) = 0$ we can write

$$D - K_\ell(y) = -\frac{c^n(\xi_n(y), y)}{y - \xi_n(y)} \stackrel{(2.13)}{=} -\frac{c_n(\xi_n(y)) - c_\ell(\xi_n(y))}{y - \xi_n(y)} - K_\ell(y)$$

and thus by (2.20)

$$D \leq 0. \quad (2.43)$$

We also have

$$l_1(y + \delta) = \delta(D - K_\ell(y)). \quad (2.44)$$

Choose $\delta \in (0, \epsilon)$ sufficiently small. Our goal is to prove $\xi_n(y + \delta) \geq \xi_n(y)$. Recall that $\xi_n(y + \delta)$ is determined from $y + \delta$ and $c^n(\cdot, y + \delta)$ only. Since we know that $\xi_n(y + \delta) \in (\xi_n(y) - \epsilon, \xi_n(y) + \epsilon) := I$ it will turn out to be enough to look at $c^n(x, y + \delta)$ only for $x \in (\xi_n(y) - \epsilon, \xi_n(y) + \epsilon)$. For such an x we have

$$c^n(x, y + \delta) - c^n(x, y) \stackrel{(2.13)}{=} (y + \delta - x)K_\ell(y + \delta) - (y - x)K_\ell(y). \quad (2.45)$$

Let l_2 be the supporting tangent to $c^n(\cdot, y + \delta) - c^n(\cdot, y)$ at $\xi_n(y)$, i.e.

$$l_2(x) = c^n(\xi_n(y), y + \delta) - c^n(\xi_n(y), y) + (x - \xi_n(y))(K_\ell(y) - K_\ell(y + \delta)).$$

Hence,

$$\begin{aligned} l_1(y + \delta) + l_2(y + \delta) &\stackrel{(2.44)}{=} \delta(D - K_\ell(y)) \\ &\quad + c^n(\xi_n(y), y + \delta) - c^n(\xi_n(y), y) \\ &\quad + (y + \delta - \xi_n(y))(K_\ell(y) - K_\ell(y + \delta)) \\ &\stackrel{(2.45)}{=} \delta D \leq 0. \end{aligned} \quad (2.46)$$

Now, since $c^n(\cdot, y + \delta) - c^n(\cdot, y)$ is linear (and therefore convex) in the domain I , $l_1 + l_2$ is a supporting tangent to $c^n(\cdot, y + \delta)$ at $\xi_n(y)$, i.e.

$$(l_1 + l_2)(x) \leq c^n(x, y + \delta) \quad \text{for } x \in I, \quad (2.47)$$

$$(l_1 + l_2)(\xi_n(y)) = c^n(\xi_n(y), y + \delta). \quad (2.48)$$

Recall that $\xi_n(y + \delta)$ is determined as the x -value where the supporting tangent to $c^n(\cdot, y + \delta)$ which passes the x -axis at $y + \delta$ touches $c^n(\cdot, y + \delta)$. Next we exploit the

fact that $\xi_n(y + \delta) \in I$ which implies that we only need to show that $\xi_n(y + \delta) \notin (\xi_n(y) - \epsilon, \xi_n(y))$. Indeed, this follows from (2.46) which yields that any supporting tangent to $c^n(\cdot, y + \delta)$ at some $\zeta \in (\xi_n(y) - \epsilon, \xi_n(y))$ must be below the x -axis when evaluated at $y + \delta$. We refer to Brown et al. [4, Fig.7] for a graphical illustration of this fact.

Now we relax the assumption (2.41). Assume that there exists a $\delta > 0$ such that $\xi_n(y) > \xi_n(y + \delta)$. We derive a contradiction to the special case as follows. Set $y_0 := y$ and $y_n := y + \delta$. Recall that ξ_n is continuous. Now we can choose $y_0 < y_1 < \dots < y_{n-1} < y_n$ such that $\xi_n(y_0) > \xi_n(y_1) > \dots > \xi_n(y_{n-1}) > \xi_n(y_n)$. Set $x_i := \xi_n(y_i)$, $i = 0, \dots, n$. Observe that by monotonicity of ξ_k , $k < n$ the graph of ξ_k intersects with at most one rectangle $(x_i, x_{i-1}) \times (y_{i-1}, y_i)$, $i = 1, \dots, n$. Consequently, there must exist at least one integer j such that the rectangle $R := (x_j, x_{j-1}) \times (y_{j-1}, y_j)$ is disjoint with the graph of every ξ_k , $k < n$. By construction and continuity of $y \mapsto \xi_n(y)$ R is not disjoint with the graph of ξ_n . Inside this rectangle R the conditions of the special case (2.41) are satisfied. Recalling that $\xi_n(y_j) = x_j < x_{j-1} = \xi_n(y_{j-1})$ and by continuity of $y \mapsto \xi_n(y)$, we can find two points $s_1 < s_2$ such that $z_1 = \xi_n(s_1) > \xi_n(s_2) = z_2$ and $(z_1, s_1) \in R, (z_2, s_2) \in R$. This is a contradiction. \square

Lemma 2.14 (ODE for K_n). *Let $n \in \mathbb{N}$ and let Assumption \otimes hold. Then*

$$y \mapsto K_n(y) \quad \text{is absolutely continuous and non-increasing.} \quad (2.49)$$

If we assume in addition that the embedding property of Theorem 2.6 is valid for the first $n - 1$ marginals then for almost all $y \geq 0$ we have:

If $\xi_n(y) < y$ then

$$K'_n(y) + \frac{K_n(y)}{y - \xi_n(y)} = K'_{J_n(y)}(y) + \frac{K_{J_n(y)}(y)}{y - \xi_n(y)} \quad (2.50)$$

where K'_j denotes the derivative of K_j which exists for almost all $y \geq 0$ and $j = 1, \dots, n$.

If $\xi_n(y) = y$ then

$$K_n(y+) = K_{J_n(y)}(y+). \quad (2.51)$$

Proof. The proof is reported in the Appendix A. \square

3 Proof of the Main Result

In this Section we prove the main result, Theorem 2.6. The key step is the identification of the distribution of the maximum, cf. Proposition 3.4.

Let $n \in \mathbb{N}$. For convenience we set

$$M_0 := 0, \quad M_i := B_{\tau_i}, \quad i = 1, \dots, n, \quad (3.1)$$

where τ_i is defined in Definition 2.2.

3.1 Basic Properties of the Embedding

Our first result shows that there is a “strong relation” between M and \bar{M} .

Lemma 3.1 (Relations Between M and \bar{M}). *Let $n \in \mathbb{N}$ and let Assumption \otimes hold. Then the following implications hold.*

$$M_n > \xi_n(y) \implies \bar{M}_n \geq y, \quad (3.2)$$

$$M_n \geq \xi_n(y) \implies \bar{M}_n \geq y \quad \text{if } \xi_n \text{ is strictly increasing at } y. \quad (3.3)$$

For $y \geq 0$ such that $j_n(y) \neq 0$ we have

$$M_{j_n(y)} \geq \xi_n(y) > \xi_{j_n(y)}(y) \implies M_n \geq \xi_n(y), \quad (3.4)$$

$$\bar{M}_{j_n(y)} < y, \quad \bar{M}_n \geq y \implies M_n \geq \xi_n(y), \quad (3.5)$$

$$\bar{M}_{j_n(y)} \geq y, \quad M_{j_n(y)} < \xi_n(y) \implies M_n < \xi_n(y). \quad (3.6)$$

If ξ_n is strictly increasing at $y \geq 0$ and $j_n(y) = 0$ then the following holds.

$$M_n \geq \xi_n(y) \iff \bar{M}_n \geq y. \quad (3.7)$$

Proof. Write $j = j_n$. We have

$$\xi_{j(y)}(y) < \xi_n(y) \leq \xi_i(y), \quad i = j(y) + 1, \dots, n.$$

In the following we are using continuity and monotonicity of ξ_1, \dots, ξ_n , cf. Lemma 2.12 and 2.13.

Case $j(y) \neq 0$. As for implication (3.2) assume that $M_n > \xi_n(y)$ and $\bar{M}_n < y$ holds. In this case M_n cannot be at the boundary ξ_n . It has to be at a boundary point $\xi_j(y')$ for some $j < n$ and some $y' < y$. However, this cannot be true because $\xi_n(y') \leq \xi_n(y) < \xi_j(y')$ and hence case (2.7a) of the definition of τ_n would have been triggered.

Implication (3.3) follows by the same arguments as for implication (3.2).

Implication (3.4) now follows from implication (3.2) applied for $j(y)$ and the fact that either $M_n = M_{j(y)}$ (case (2.7b)) or M moves to a point at the boundary $\xi_i(y') \geq \xi_n(y)$ for some $i = j(y) + 1, \dots, n$, $y' \geq y$ (case (2.7a)).

Implication (3.5) holds because if M increases its maximum at time $j(y)$, which is $< y$, to some $y' \geq y$ at time n , it will hit a boundary point $\xi_i(y') \geq \xi_n(y)$ for some $i = j(y) + 1, \dots, n$.

Implication (3.6) holds because from $\bar{M}_{j(y)} \geq y$ and $M_{j(y)} < \xi_n(y)$ it follows that $M_{j(y)} = \xi_i(y') < \xi_n(y) \leq \xi_j(y')$ for some $i \leq j(y)$, $y' \geq y$, $j > j(y)$. From this it follows that M will stay where it is until time n , cf. case (2.7b).

Case $j(y) = 0$. The condition $M_n \geq \xi_n(y)$ implies in a similar fashion as in (3.3) that $\bar{M}_n \geq y$ holds. Conversely, assume that $\bar{M}_n \geq y$ holds. In this case M_n must be at a boundary point $\xi_i(y') \geq \xi_n(y)$ for some $i = 1, \dots, n$, $y' \geq y$. \square

As an application of Lemma 3.1 we obtain the following result.

Lemma 3.2 (Contributions to the Maximum). *Let $n \in \mathbb{N}$ and let Assumption \otimes hold. Assume ξ_n is strictly increasing at $y \geq 0$.*

Then, if $j_n(y) \neq 0$

$$\mathbb{P}[\bar{M}_n \geq y] = \mathbb{P}[M_n \geq \xi_n(y)] - \mathbb{P}[M_{j_n(y)} \geq \xi_n(y)] + \mathbb{P}[\bar{M}_{j_n(y)} \geq y] \quad (3.8)$$

and if $j_n(y) = 0$

$$\mathbb{P}[\bar{M}_n \geq y] = \mathbb{P}[M_n \geq \xi_n(y)]. \quad (3.9)$$

Proof. Write $j = j_n$.

Case $j(y) \neq 0$. Firstly, let us compute

$$\begin{aligned} & \mathbb{P}[\bar{M}_n \geq y] - \mathbb{P}[M_n \geq \xi_n(y)] \\ \stackrel{(3.3)}{=} & \mathbb{P}[\bar{M}_n \geq y] - \mathbb{P}[M_n \geq \xi_n(y), \bar{M}_n \geq y] = \mathbb{P}[\bar{M}_n \geq y, M_n < \xi_n(y)] \\ = & \mathbb{P}[\bar{M}_n \geq y, M_n < \xi_n(y), \bar{M}_{j(y)} \geq y] + \mathbb{P}[\bar{M}_n \geq y, M_n < \xi_n(y), \bar{M}_{j(y)} < y] \\ \stackrel{(3.4)}{=} & \mathbb{P}[M_n < \xi_n(y), \bar{M}_{j(y)} \geq y, M_{j(y)} < \xi_n(y)] \\ \stackrel{(3.5)}{=} & \mathbb{P}[M_n < \xi_n(y), \bar{M}_{j(y)} \geq y, M_{j(y)} < \xi_n(y)]. \end{aligned}$$

Secondly, let us compute

$$\begin{aligned}
& \mathbb{P} [\bar{M}_{j(y)} \geq y] - \mathbb{P} [M_{j(y)} \geq \xi_n(y)] \\
&= \mathbb{P} [\bar{M}_{j(y)} \geq y, M_{j(y)} \geq \xi_n(y)] + \mathbb{P} [\bar{M}_{j(y)} \geq y, M_{j(y)} < \xi_n(y)] - \mathbb{P} [M_{j(y)} \geq \xi_n(y)] \\
&\stackrel{(3.2)}{=} \mathbb{P} [\bar{M}_{j(y)} \geq y, M_{j(y)} < \xi_n(y)] \stackrel{(3.6)}{=} \mathbb{P} [M_n < \xi_n(y), \bar{M}_{j(y)} \geq y, M_{j(y)} < \xi_n(y)].
\end{aligned}$$

Comparing these two equations yields the claim.

Case $j(y) = 0$. The claim follows from (3.7). \square

3.2 Law of the Maximum

Our next goal is to identify the distribution of M_n . We will achieve this by deriving an ODE for $\mathbb{P} [\bar{M}_n \geq \cdot]$ using excursion theoretical results, cf. Lemma 3.3, and link it to the ODE satisfied by K_n , cf. Lemma 2.14.

Lemma 3.3 (ODE for the Maximum). *Let $n \in \mathbb{N}$ and let Assumption \circledast hold. Then the mapping*

$$y \mapsto \mathbb{P} [\bar{M}_n \geq y]$$

is absolutely continuous and for almost all $y \geq 0$ we have:

If $\xi_n(y) < y$ then

$$\frac{\partial \mathbb{P} [\bar{M}_n \geq y]}{\partial y} + \frac{\mathbb{P} [\bar{M}_n \geq y]}{y - \xi_n(y)} = \frac{\mathbb{P} [\bar{M}_{j_n(y)} \geq y]}{y - \xi_n(y)} + \frac{\partial \mathbb{P} [\bar{M}_j \geq y]}{\partial y} \Bigg|_{j=j_n(y)}. \quad (3.10)$$

If $\xi_n(y) = y$ then

$$\mathbb{P} [\bar{M}_n > y] = \mathbb{P} [\bar{M}_{j_n(y)} > y]. \quad (3.11)$$

Proof. Write $j = j_n$. We exclude all $y > 0$ which are a discontinuity of ξ_1 . This is clearly a null-set.

The cases $n = 1, 2$ are true by Brown et al. [4]. Assume by induction hypothesis that we have proven the claim for $i = 1, \dots, n-1$.

If $\xi_n(y) = y$ then it is clear from the definition of the embedding, cf. Definition 2.2, that

$$\bar{M}_n > y \iff \bar{M}_j > y. \quad (3.12)$$

Case $j(y) \neq 0$. For $\delta > 0$ we have

$$\begin{aligned}
& \mathbb{P} [\bar{M}_n \geq y + \delta, \bar{M}_{j(y)} < y + \delta] - \mathbb{P} [\bar{M}_n \geq y, \bar{M}_{j(y)} < y] \\
&= \mathbb{P} [\bar{M}_n \geq y + \delta, \bar{M}_{j(y)} < y] - \mathbb{P} [\bar{M}_n \geq y, \bar{M}_{j(y)} < y] \\
&+ \underbrace{\mathbb{P} [\bar{M}_n \geq y + \delta, y < \bar{M}_{j(y)} < y + \delta]}_{=0 \text{ for } \delta > 0 \text{ small enough by definition of } j(y) \text{ and continuity of } \xi_i}.
\end{aligned} \quad (3.13)$$

For $r > 0$ we define

$$\begin{aligned}
\bar{\xi}_j(r) &:= \max_{k: j \leq k \leq n} \{\xi_k(r) : \xi_k(y) = \xi_n(y)\}, \\
\underline{\xi}_j(r) &:= \min_{k: j \leq k \leq n} \{\xi_k(r) : \xi_k(y) = \xi_n(y)\}
\end{aligned}$$

and note that

$$\bar{\xi}_{j(y)}(r) \rightarrow \xi_n(y), \quad \underline{\xi}_{j(y)}(r) \rightarrow \xi_n(y) \quad \text{as } r \rightarrow y \quad (3.14)$$

by continuity of ξ_i at y for $i = 1, \dots, n$.

Let $\delta > 0$. We have by excursion theoretical results, cf. e.g. Rogers [19],

$$\begin{aligned}
& \mathbb{P} [\bar{M}_n \geq y, \bar{M}_{j(y)} < y] \exp \left(- \int_y^{y+\delta} \frac{dr}{r - \bar{\xi}_{j(y)}(r)} \right) \\
& \leq \mathbb{P} [\bar{M}_n \geq y + \delta, \bar{M}_{j(y)} < y] \\
& \leq \mathbb{P} [\bar{M}_n \geq y, \bar{M}_{j(y)} < y] \exp \left(- \int_y^{y+\delta} \frac{dr}{r - \underline{\xi}_{j(y)}(r)} \right).
\end{aligned} \tag{3.15}$$

Now we compute for y such that $\xi_n(y) < y$

$$\begin{aligned}
& \frac{\mathbb{P} [\bar{M}_n \geq y + \delta, \bar{M}_{j(y)} < y + \delta] - \mathbb{P} [\bar{M}_n \geq y, \bar{M}_{j(y)} < y]}{\delta} \\
& \stackrel{(3.13), (3.15)}{\leq} \mathbb{P} [\bar{M}_n \geq y, \bar{M}_{j(y)} < y] \frac{\exp \left(- \int_y^{y+\delta} \frac{dr}{r - \underline{\xi}_{j(y)}(r)} \right) - 1}{\delta} \\
& \xrightarrow[\text{as } \delta \downarrow 0]{\text{by (3.14)}} - \frac{\mathbb{P} [\bar{M}_n \geq y, \bar{M}_{j(y)} < y]}{y - \xi_n(y)}
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
& \frac{\mathbb{P} [\bar{M}_n \geq y + \delta, \bar{M}_{j(y)} < y + \delta] - \mathbb{P} [\bar{M}_n \geq y, \bar{M}_{j(y)} < y]}{\delta} \\
& \stackrel{(3.13), (3.15)}{\geq} \mathbb{P} [\bar{M}_n \geq y, \bar{M}_{j(y)} < y] \frac{\exp \left(- \int_y^{y+\delta} \frac{dr}{r - \xi_{j(y)}(r)} \right) - 1}{\delta} \\
& \xrightarrow[\text{as } \delta \downarrow 0]{\text{by (3.14)}} - \frac{\mathbb{P} [\bar{M}_n \geq y, \bar{M}_{j(y)} < y]}{y - \xi_n(y)}.
\end{aligned} \tag{3.17}$$

Hence, from (3.16) and (3.17) it follows that the right-derivative of

$$y \mapsto \mathbb{P} [\bar{M}_n \geq y, \bar{M}_j < y] \Big|_{j=j(y)} \tag{3.18}$$

exists. Similar arguments for $\delta < 0$ show that the left-derivative exists and is the same as the right-derivative. Local Lipschitz continuity of (3.18) then follows from (3.16) and (3.17).

Observe the obvious equality

$$\mathbb{P} [\bar{M}_n \geq y] = \mathbb{P} [\bar{M}_j \geq y] + \mathbb{P} [\bar{M}_n \geq y, \bar{M}_j < y] \tag{3.19}$$

Taking $j = j(y)$ in (3.19) and fixing it, we conclude by induction hypothesis that $y \mapsto \mathbb{P} [\bar{M}_n > y]$ is locally Lipschitz continuous and hence absolutely continuous and its derivative reads

$$\frac{\partial \mathbb{P} [\bar{M}_n \geq y]}{\partial y} = \frac{\partial \mathbb{P} [\bar{M}_j \geq y]}{\partial y} \Big|_{j=j_n(y)} + \frac{\mathbb{P} [\bar{M}_{j_n(y)} \geq y] - \mathbb{P} [\bar{M}_n \geq y]}{y - \xi_n(y)}.$$

Case $j(y) = 0$. For $\delta > 0$ we have by excursion theoretical results

$$\begin{aligned}
& \mathbb{P} [\bar{M}_n \geq y + \delta, \bar{M}_1 < y + \delta] - \mathbb{P} [\bar{M}_n \geq y, \bar{M}_1 < y] \\
= & \mathbb{P} [\bar{M}_n \geq y + \delta, \bar{M}_1 < y + \delta] - \mathbb{P} [\bar{M}_n \geq y + \delta, \bar{M}_1 < y] \\
& + \mathbb{P} [\bar{M}_n \geq y + \delta, \bar{M}_1 < y] - \mathbb{P} [\bar{M}_n \geq y, \bar{M}_1 < y] \\
\leq & \int_y^{y+\delta} \mathbb{P} [\bar{M}_1 \in ds] \frac{(\xi_1(s) - \bar{\xi}_1(s))^+}{s - \bar{\xi}_1(s)} \exp \left(- \int_s^{y+\delta} \frac{dr}{r - \bar{\xi}_1(r)} \right) \\
& + \mathbb{P} [\bar{M}_n \geq y, \bar{M}_1 < y] \left[\exp \left(- \int_y^{y+\delta} \frac{dr}{r - \bar{\xi}_1(r)} \right) - 1 \right]. \tag{3.20}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \mathbb{P} [\bar{M}_n \geq y + \delta, \bar{M}_1 < y + \delta] - \mathbb{P} [\bar{M}_n \geq y, \bar{M}_1 < y] \\
\geq & \int_y^{y+\delta} \mathbb{P} [\bar{M}_1 \in ds] \frac{(\xi_1(s) - \bar{\xi}_1(s))^+}{s - \bar{\xi}_1(s)} \exp \left(- \int_s^{y+\delta} \frac{dr}{r - \bar{\xi}_1(r)} \right) \\
& + \mathbb{P} [\bar{M}_n \geq y, \bar{M}_1 < y] \left[\exp \left(- \int_y^{y+\delta} \frac{dr}{r - \bar{\xi}_1(r)} \right) - 1 \right]. \tag{3.21}
\end{aligned}$$

From (3.20) and (3.21) it follows that the right-derivative of

$$y \mapsto \mathbb{P} [\bar{M}_n \geq y, \bar{M}_1 < y] \tag{3.22}$$

exists. Similar arguments for $\delta < 0$ show that the left-derivative exists and is the same as the right-derivative. Local Lipschitz continuity of (3.22) then follows from (3.20) and (3.21). Now we can conclude from (3.19)–(3.21) applied with $j = 1$ that $y \mapsto \mathbb{P} [\bar{M}_n \geq y]$ is locally Lipschitz continuous and hence absolutely continuous and its derivative reads

$$\begin{aligned}
& \frac{\partial \mathbb{P} [\bar{M}_n \geq y]}{\partial y} \\
\stackrel{(3.14)}{=} & \frac{\partial \mathbb{P} [\bar{M}_1 \geq y]}{\partial y} - \frac{\partial \mathbb{P} [\bar{M}_1 \geq y]}{\partial y} \frac{(\xi_1(y) - \xi_n(y))^+}{y - \xi_n(y)} - \frac{\mathbb{P} [\bar{M}_n \geq y] - \mathbb{P} [\bar{M}_1 \geq y]}{y - \xi_n(y)},
\end{aligned}$$

which implies by induction hypothesis

$$\frac{\mathbb{P} [\bar{M}_n \geq y]}{y - \xi_n(y)} + \frac{\partial \mathbb{P} [\bar{M}_n \geq y]}{\partial y} = 0.$$

This finishes the proof. \square

Finally, we argue that $\mathbb{P} [\bar{M}_n \geq y] = K_n(y)$ holds for all $y \geq 0$.

Proposition 3.4 (Law of the Maximum). *Let $n \in \mathbb{N}$ and let Assumption \otimes hold. Assume that the embedding property of Theorem 2.6 is valid for the first $n - 1$ marginals. Then for all $y \geq 0$ we have*

$$\mathbb{P} [\bar{M}_n \geq y] = K_n(y). \tag{3.23}$$

Proof. The case $n = 1$ holds by the Azéma-Yor embedding. Assume by induction hypothesis that

$$K_i = \mathbb{P} [\bar{M}_i \geq \cdot], \quad i = 1, \dots, n - 1.$$

In Lemma 2.14 and 3.3 we derived an ODE for K_n and $\mathbb{P}[\bar{M}_n \geq \cdot]$, respectively, in terms of K_1, \dots, K_{n-1} and $\mathbb{P}[\bar{M}_1 \geq \cdot], \dots, \mathbb{P}[\bar{M}_{n-1} \geq \cdot]$, respectively. These ODEs are valid for a.e. $y \geq 0$. By induction hypothesis both drivers of these ODEs coincide everywhere and hence the claim follows from the boundary conditions

$$\begin{aligned} K_n(y) &\rightarrow 0 \quad \text{as } y \rightarrow \infty, & K_n(y) &\rightarrow 1 \quad \text{as } y \rightarrow 0, \\ \mathbb{P}[\bar{M}_n \geq y] &\rightarrow 0 \quad \text{as } y \rightarrow \infty, & \mathbb{P}[\bar{M}_n \geq y] &\rightarrow 1 \quad \text{as } y \rightarrow 0, \end{aligned}$$

absolute continuity of K_n and $\mathbb{P}[\bar{M}_n \geq \cdot]$ and the fact that the ODE

$$\left(\mathbb{P}[\bar{M}_n \geq y] - K_n(y) \right)' = -\frac{\mathbb{P}[\bar{M}_n \geq y] - K_n(y)}{y - \xi_n(y)}, \quad \mathbb{P}[\bar{M}_n \geq 0] - K_n(0) = 0,$$

has unique solution given by 0. \square

3.3 Embedding Property

In this subsection we prove that the stopping times τ_1, \dots, τ_n from Definition 2.2 embed the laws μ_1, \dots, μ_n if Assumption \otimes is in place. More precisely, given Proposition 3.4 above and by inductive reasoning, to complete the proof of Theorem 2.6 we only need to show the following:

Proposition 3.5 (Embedding). *In the setup of Theorem 2.6 we have*

$$B_{\tau_n} \sim \mu_n \tag{3.24}$$

and $(B_{\tau_n \wedge t})_{t \geq 0}$ is a uniformly integrable martingale.

Proof. The case $n = 1$ is just the Azéma-Yor embedding. By induction hypothesis, assume that the claim holds for all $i \leq n - 1$.

We claim that ξ_n ranges continuously over the full support of μ_n . This is because, firstly, we know from Lemma 2.13 that ξ_2, \dots, ξ_n are continuous. Secondly, we have by using $l_{\mu_n} \leq l_{\mu_i}$ that

$$\inf_{\zeta \leq 0} \frac{c^n(\zeta, 0)}{0 - \zeta} \geq \inf_{\zeta \leq 0} \min_{1 \leq i < n} \left\{ \underbrace{\frac{c_n(\zeta) - c_i(\zeta)}{0 - \zeta}}_{\geq 0} + \underbrace{K_i(0)}_{=1} \right\} \wedge \underbrace{\frac{c_n(l_{\mu_n})}{0 - l_{\mu_n}}}_{=1} = 1$$

which shows that $\xi_n(0) = l_{\mu_n}$. Furthermore, by using $r_{\mu_n} \geq r_{\mu_i}$ we have from (2.11) and (2.17) that

$$\xi_n(r_{\mu_n}) = r_{\mu_n}.$$

Let $y > 0$ be such that ξ_n is differentiable and strictly increasing at y , $\xi_n(y)$ is not an atom of neither μ_n nor $\mu_{J_n(y)}$ and y is not a discontinuity of ξ_1 . Note that for such a y equation (A.6) holds because of (A.7). Applying previous results we obtain

$$\begin{aligned} &\mathbb{P}[M_n \geq \xi_n(y)] - \mathbb{P}[M_{J_n(y)} \geq \xi_n(y)] + \mathbb{P}[\bar{M}_{J_n(y)} \geq y] \\ \stackrel{\text{Lemma 3.2}}{=} &\mathbb{P}[\bar{M}_n \geq y] \\ \stackrel{\text{Prop. 3.4}}{=} &K_n(y) \\ \stackrel{\text{(A.6)}}{=} &-c'_n(\xi_n(y)) + c'_{J_n(y)}(\xi_n(y)) + K_{J_n(y)}(y), \end{aligned}$$

which implies by induction hypothesis that

$$\mathbb{P}[M_n \geq \xi_n(y)] = -c'_n(\xi_n(y)) = \mu_n([\xi_n(y), \infty)).$$

We have matched the distribution of M_n to μ_n at almost all points inside the support. The embedding property follows.

Now we prove uniform integrability by applying a result from Azéma et al. [2] which states that if

$$\lim_{x \rightarrow \infty} x \mathbb{P} [\bar{B}|_{\tau_n} \geq x] = 0 \quad (3.25)$$

then $(B_{\tau_n \wedge t})_{t \geq 0}$ is uniformly integrable.

Let us verify (3.25). Set $H_x = \inf \{t > 0 : B_t = x\}$. We have (here ξ_i^{-1} denotes the left-continuous inverse of ξ_i)

$$\begin{aligned} \mathbb{P} [\bar{B}|_{\tau_n} \geq x] &\leq \mathbb{P} [H_{-x} < H_{\max_{i \leq n} \xi_i^{-1}(-x)}] + \mathbb{P} [\bar{B}_{\tau_n} \geq x] \\ &= \frac{\max_{i \leq n} \xi_i^{-1}(-x)}{x + \max_{i \leq n} \xi_i^{-1}(-x)} + K_n(x). \end{aligned}$$

From the definition of ξ_n , cf. (2.4), and the properties of b_i , cf. (2.11) we have

$$0 \leq \max_{i \leq n} \xi_i^{-1}(-x) \leq \max_{i \leq n} b_i(-x) \xrightarrow{x \rightarrow \infty} 0$$

and hence, recalling the definition of μ_n^{HL} in (2.24),

$$\lim_{x \rightarrow \infty} x \mathbb{P} [\bar{B}|_{\tau_n} \geq x] \leq \lim_{x \rightarrow \infty} x K_n(x) \leq \lim_{x \rightarrow \infty} x \frac{c_n(b_n^{-1}(x))}{x - b_n^{-1}(x)} = \lim_{x \rightarrow \infty} x \mu_n^{\text{HL}}([x, \infty)) = 0.$$

This finishes the proof. \square

4 Discussion of Assumption \otimes and Extensions

In this section we focus on our main technical assumption so far: the condition (ii) in Assumption \otimes . We construct a simple example of probability measures μ_1, μ_2, μ_3 which violate the condition and where the stopping boundaries ξ_1, ξ_2, ξ_3 , obtained via (2.4), fail to embed (μ_1, μ_2, μ_3) . It follows that the assumption is not merely technical but does rule out certain type of interdependence between the marginals. If it is not satisfied then it may not be enough to perturb the measures slightly to satisfy it.

We then present an extension of our embedding, in the case $n = 3$, which works in all generality. More precisely, we show how to modify the optimisation problem from which ξ_3 is determined in order to obtain the embedding property. The general embedding, as compared to the embedding in the presence of Assumption \otimes (ii), gains an important degree of freedom and becomes less explicit. In consequence it is also much harder to implement in practice, to the point that we do not believe this is worth pursuing for $n > 3$. This is also why, as well as for the sake of brevity, we keep the discussion in the section rather formal.

4.1 Counterexample for Assumption \otimes (ii)

In Figure 4.1 we define measures via their potentials

$$U\mu : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto U\mu(x) := - \int_{\mathbb{R}} |u - x| \mu(du). \quad (4.1)$$

We refer to Obłój [17, Proposition 2.3] for useful properties of $U\mu$.

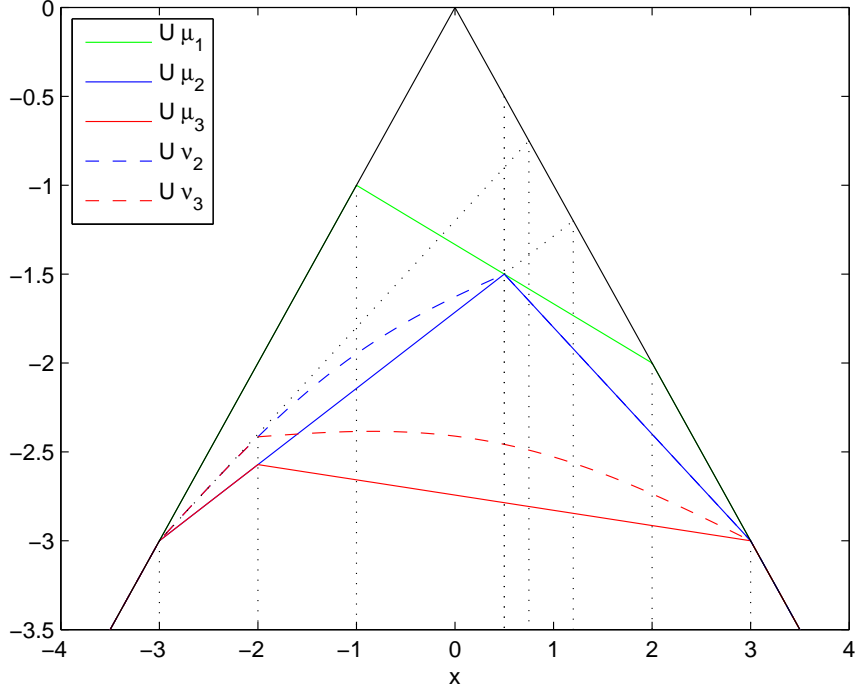


Figure 4.1: Potentials of $\mu_1, \mu_2, \mu_3, \nu_2$ and ν_3 .

The measures with potentials illustrated in Figure 4.1 are given as

$$\mu_1(\{-1\}) = \frac{2}{3}, \quad \mu_1(\{2\}) = \frac{1}{3}, \quad (4.2)$$

$$\mu_2(\{-3\}) = \frac{2}{7}, \quad \mu_2\left(\left\{\frac{1}{2}\right\}\right) = \frac{18}{35}, \quad \mu_2(\{3\}) = \frac{1}{5}, \quad (4.3)$$

$$\mu_3(\{-3\}) = \frac{2}{7}, \quad \mu_3(\{-2\}) = \frac{9}{35}, \quad \mu_3(\{3\}) = \frac{16}{35}. \quad (4.4)$$

Observe that the embedding for (μ_1, μ_2, μ_3) is unique: We write $H_{a,b}$ for the exit time of $[a, b]$ and denote $H_{a,b} \circ \theta_\tau := \inf\{t > \tau : B_t \notin (a, b)\}$. Then the embedding $(\tau_1, \tau'_2, \tau_3)$ can be written as

$$\tau_1 = H_{-1,2}, \quad \tau'_2 = H_{-3, \frac{1}{2}} \circ \theta_{\tau_1} \mathbb{1}_{\{B_{\tau_1} = -1\}} + H_{\frac{1}{2}, 3} \circ \theta_{\tau_1} \mathbb{1}_{\{B_{\tau_1} = 2\}}, \quad \tau_3 = H_{-2,3} \circ \theta_{\tau_2}. \quad (4.5)$$

As mentioned earlier, our construction yields the same first two stopping boundaries as the method of Brown et al. [4]. In this case, cf. Figure 4.2,

$$\xi_1(y) := \begin{cases} -1 & \text{if } y \in [0, 2), \\ y & \text{else,} \end{cases} \quad \xi_2(y) := \begin{cases} -3 & \text{if } y \in [0, \frac{1}{2}), \\ \frac{1}{2} & \text{if } y \in [\frac{1}{2}, 3), \\ y & \text{else.} \end{cases}$$

This already shows that our embedding fails to embed μ_2 . To see this one just has to compare the stopping boundary ξ_2 in the Definition of τ_2 with (4.5). In Section 4.2 we will recall from Brown et al. [4] how the stopping time τ_2 has to be modified into τ'_2 , giving the stopping time above.

More importantly, the embedding for μ_3 fails because the optimization problem (2.4) does not return the third (unique) stopping boundary which is required for the embedding of (μ_1, μ_2, μ_3) . Indeed, for sufficiently small $y > \frac{1}{2}$, in the region $\zeta < \min(\xi_1(y), \xi_2(y)) = -1$ we are looking at the minimization of $\zeta \mapsto \frac{c_3(\zeta)}{y-\zeta}$ which is attained by $\xi_3(y) = -3 < -2$ since μ_3 has an atom at -3 . Consequently, there will be a positive probability to hit -3 after τ_2 . This contradicts (4.4). This, together with the correct third boundary ξ_3 , is illustrated in Figure 4.2.

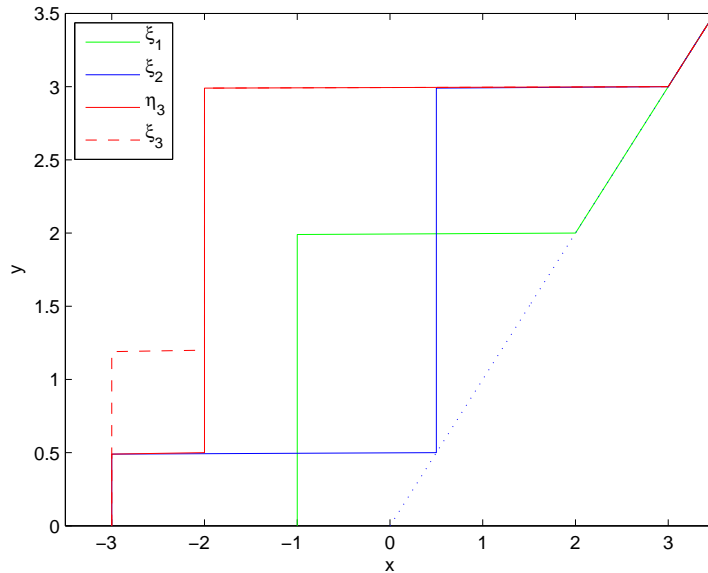


Figure 4.2: We illustrate the (unique) boundaries ξ_1, ξ_2, η_3 required for the embedding of (μ_1, μ_2, μ_3) from (4.2)–(4.4) and the stopping boundary ξ_3 obtained from (2.4). In order to ensure the embedding for μ_2 , the mass stopped at τ_2 in -1 on the event $\{\bar{B}_{\tau_2} \in (1/2, 2)\}$ is diffused to -3 or to $1/2$ at τ'_2 , without affecting the maximum: $\bar{B}_{\tau_2} = \bar{B}_{\tau'_2}$. Note that the case $\xi_2(y) = y$, here for $y = 1/2$, is possible and required to define the embedding. After τ'_2 we need to define τ_3 which embeds μ_3 which here is implied directly by (4.5). In Section 4.2 we develop arguments which generalise this.

This example does not contradict our main result because Assumption $\textcircled{*}(\text{ii})(\text{a})$ is not satisfied for $i = 2$ and $y = \frac{1}{2}$, where $\zeta = -3$ minimizes the objective function but $c_2(\frac{1}{2}) = c_1(\frac{1}{2})$ holds. Our counterexample also shows that a “small perturbation” to (μ_1, μ_2, μ_3) does not remove the problem. Indeed, similar reasoning to the one above holds for measures (μ_1, ν_2, ν_3) defined by their potentials in Figure 4.1. Assumption $\textcircled{*}$ rules out certain type of subtle structure between the marginals and not only some “isolated” or “singular” configurations of measures.

4.2 Sketch for General Embedding in the Case $n = 3$

In the example of the measures (μ_1, μ_2, μ_3) from (4.2)–(4.4) the (unique) embedding could still be seen as a type of “iterated Azéma-Yor type embedding” although it does not satisfy the relations from Lemma 3.1. Consequently, one might conjecture that a modification of the optimization problem (2.4) and a relaxation of Lemma 3.1 might lead to a generally applicable embedding. We now explain in which sense

this is true. Our aim is to outline new ideas and arguments which are needed. The technical details quickly become very involved and lengthy. In the sake of brevity, but also to better illustrate the main points, we restrict ourselves to a formal discussion and the case $n = 3$.

In order to understand the problem in more detail, we need to recall from Brown et al. [4] how the embedding for μ_2 looks like in general. It reads

$$\tau_2^{\text{BHR}} := \begin{cases} \tau'_2 & \text{if } \xi_2^{-1} \neq \xi_1(\bar{B}_{\tau_1}) \text{ and } \xi_1(\bar{B}_{\tau_1}) < \xi_2(\bar{B}_{\tau_1}), \\ \tau_2 & \text{else,} \end{cases} \quad (4.6)$$

where τ'_2 is some stopping time with $\bar{B}_{\tau_2} = \bar{B}_{\tau'_2}$. Its existence is established by Brown et al. [4] by showing that the relative parts of the mass which are further diffused have the same mass, mean and are in convex order. In general there will be infinitely many such stopping times τ'_2 . Although this is not true for (μ_1, μ_2, μ_3) in (4.2)–(4.4) because their embedding was unique, it is true for measures (μ_1, ν_2, ν_3) which are defined via their potentials in Figure 4.1.

Let ξ_1 and ξ_2 be defined as in (2.4) and let $M_2 = B_{\tau_2}$. Now our goal is to define an embedding $\tilde{\tau}_3$ for the third marginal on top of the embedding of Brown et al. [4] in a situation as in (μ_1, ν_2, ν_3) . We still want to define our iterated Azéma-Yor type embedding through a stopping rule based on some stopping boundary $\tilde{\xi}_3$ as a first exit time,

$$\tilde{\tau}_3 := \begin{cases} \inf_{\tau_2^{\text{BHR}}} \left\{ t \geq \tau_2^{\text{BHR}} : B_t \leq \tilde{\xi}_3(\bar{B}_t) \right\} & \text{if } B_{\tau_2^{\text{BHR}}} > \tilde{\xi}_3(\bar{B}_{\tau_2^{\text{BHR}}}), \\ \tau_2^{\text{BHR}} & \text{else,} \end{cases} \quad (4.7)$$

and prove that this is a valid embedding of μ_3 . We observe that now the choice of τ'_2 in the definition of τ_2^{BHR} may matter for the subsequent embedding. Similarly as in the embedding of Brown et al. [4] we expect that this will be only possible if the procedure which produces $\tilde{\xi}_3$ yields a continuous $\tilde{\xi}_3$. Otherwise an additional step, producing a stopping time $\tau'_3 \geq \tilde{\tau}_3$ would be required and further complicate the presentation.

With this, a more canonical approach in the context of Lemma 3.1 is to write

$$\mathbb{P}[\bar{M}_3 \geq y] = \mathbb{P}[M_3 \geq \tilde{\xi}_3(y)] + \text{“error-term”}, \quad (4.8)$$

which we formalize in (4.27). As it will turn out, this “error-term” provides a suitable “book-keeping procedure” to keep track of the masses in the embedding. We proceed along the lines of the proof of our main result. For simplicity, we further assume that ξ_2 has only one discontinuity, i.e. $\underline{z} := \xi_2(\underline{y}-) < \xi_2(\underline{y}+) := \bar{z}$ for some $\underline{y} \geq 0$ and we let $\bar{y} := \xi_1^{-1}(\bar{z})$. As explained below, this is not restrictive since our procedure is localised. If $\bar{y} \leq \underline{y}$ then μ_1 can be “ignored” and the results of Brown et al. [4] apply. Hence we assume $\bar{y} > \underline{y}$.

4.2.1 Redefining ξ_3 and K_3

Define the following auxiliary terms,

$$F(\zeta, y; \tau'_2) := \mathbb{1}_{\{\bar{M}_1 \geq y\}} (\zeta - M_2)^+, \quad (4.9)$$

$$f^{\text{iAY}}(\zeta, y; \tau'_2) := \mathbb{E}[F(\zeta, y; \tau'_2)]. \quad (4.10)$$

As the notation underlines, these quantities may depend on the additional choice of stopping time τ'_2 between τ_2 and τ_3 . Note that for $\zeta \in [\underline{z}, \bar{z}]$ and $y \in [\underline{y}, \bar{y}]$,

$$\frac{\partial f^{\text{iAY}}}{\partial \zeta}(\zeta, y; \tau'_2) = \mathbb{P}[\bar{M}_1 \geq y, M_2 < \zeta], \quad (4.11)$$

and

$$\begin{aligned} \frac{\partial f^{\text{iAY}}}{\partial y}(\zeta, y; \tau'_2) &= -\mathbb{E} \left[\frac{\mathbb{1}_{\{\bar{M}_1 \in dy, M_2 < \zeta\}}}{dy} \right] \zeta + \mathbb{E} \left[\frac{\mathbb{1}_{\{\bar{M}_1 \in dy, M_2 < \zeta\}}}{dy} M_2 \right] \\ &= -\left(\zeta - \alpha(\zeta, y; \tau'_2)\right) \frac{\mathbb{P}[\bar{M}_1 \in dy, M_2 < \zeta]}{dy} \end{aligned} \quad (4.12)$$

where

$$\alpha(\zeta, y; \tau'_2) := \mathbb{E} [M_2 | \bar{M}_1 = y, M_2 < \zeta], \quad (4.13)$$

$$\beta(\zeta, y; \tau'_2) := \mathbb{E} [M_2 | \bar{M}_1 = y, M_2 \geq \zeta]. \quad (4.14)$$

With these definitions we have by the properties of τ'_2 ,

$$\begin{aligned} \alpha(\zeta, y; \tau'_2) \frac{\mathbb{P}[\bar{M}_1 \in dy, M_2 < \zeta]}{dy} + \beta(\zeta, y; \tau'_2) \frac{\mathbb{P}[\bar{M}_1 \in dy, M_2 \geq \zeta]}{dy} \\ = \xi_1(y) \frac{\mathbb{P}[\bar{M}_1 \in dy]}{dy}. \end{aligned} \quad (4.15)$$

We now redefine ξ_3 and K_3 from (2.4) and (2.6), respectively, and denote the new definition by $\tilde{\xi}_3$ and \tilde{K}_3 . To this end, introduce the function

$$\tilde{c}^3(\zeta, y) := \begin{cases} c_3(\zeta) - f^{\text{iAY}}(\zeta, y; \tau'_2) & \text{if } \underline{z} \leq \zeta \leq \bar{z}, \underline{y} \leq y \leq \bar{y}, \\ c^3(\zeta, y) & \text{else.} \end{cases} \quad (4.16a)$$

$$\quad (4.16b)$$

We have that \tilde{c}^3 is continuous and $\tilde{c}^3 \leq c^3$. Using the properties of τ'_2 this can be seen from the following:

$$\begin{aligned} f^{\text{iAY}}(\zeta, y; \tau'_2) &= \mathbb{E} \left[(\zeta - M_2)^+ \mathbb{1}_{\{\bar{M}_1 \geq y\}} \right] = \mathbb{E} \left[\left\{ (M_2 - \zeta)^+ - (y - \zeta) \right\} \mathbb{1}_{\{\bar{M}_1 \geq y\}} \right] \\ &= \mathbb{E} \left[(M_2 - \zeta)^+ \mathbb{1}_{\{\bar{M}_1 \geq y\}} \right] - (y - \zeta) K_1(y) \\ &\geq \begin{cases} c_1(\zeta) - (y - \zeta) K_1(y) & \text{if } \zeta > \xi_1(y), \\ 0 & \text{else.} \end{cases} \end{aligned} \quad (4.17)$$

for $\zeta \in [\underline{z}, \bar{z}]$ and $y \in [\underline{y}, \bar{y}]$, with equality for $\zeta = \bar{z}$. Continuity at $\zeta = \underline{z}$ holds by the properties of τ'_2 . For $y = \bar{y}$ we have $f^{\text{iAY}}(\zeta, \bar{y}; \tau'_2) = 0$. As for continuity at $y = \underline{y}$ it is enough to observe

$$\mathbb{E} \left[(M_2 - \zeta)^+ \mathbb{1}_{\{\bar{M}_1 \geq y\}} \right] = c_2(\zeta) - (y - \zeta)(K_2(y) - K_1(y)).$$

As before, let

$$\tilde{\xi}_3(y) := \arg \min_{\zeta < y} \frac{\tilde{c}^3(\zeta, y)}{y - \zeta} \quad (4.18)$$

and

$$\tilde{K}_3(y) := \frac{\tilde{c}^3(\tilde{\xi}_3(y), y)}{y - \tilde{\xi}_3(y)}. \quad (4.19)$$

It is clear that a discontinuity of ξ_2 results in a local perturbation of c^3 into \tilde{c}^3 and in consequence of ξ_3 into $\tilde{\xi}_3$. If ξ_2 has multiple discontinuities the construction above applies to each of them giving a global definition of \tilde{c}^3 . Then \tilde{K}_3 and $\tilde{\xi}_3$ are defined as above.

4.2.2 Law of the Maximum

In the following we assume that $\zeta \in [\underline{z}, \bar{z}]$ and $y \in [\underline{y}, \bar{y}]$. Otherwise $\tilde{c}^3 = c^3$ and the arguments from Sections 2 and 3 apply. We have $\tilde{\xi}_3(y) < \xi_2(y)$ and $\bar{M}_1 = \bar{M}_2$ on $\{\bar{M}_1 \in [\underline{y}, \bar{y}]\}$.

Note the obvious decomposition

$$\mathbb{P}[\bar{M}_3 \geq y] = \mathbb{P}[\bar{M}_1 < y, \bar{M}_3 \geq y] + \mathbb{P}[\bar{M}_1 \geq y].$$

We compute by similar excursion theoretical arguments as in the proof of Lemma 3.3,

$$\begin{aligned} & \left. \frac{\partial \mathbb{P}[\bar{M}_1 < y, \bar{M}_3 \geq m]}{\partial y} \right|_{m=y} =: p(\tilde{\xi}_3(y), y; \tau'_2) \\ & = \frac{\mathbb{P}[M_2 > \tilde{\xi}_3(y), \bar{M}_1 \in dy]}{dy} \frac{\beta(\tilde{\xi}_3(y), y; \tau'_2) - \tilde{\xi}_3(y)}{y - \tilde{\xi}_3(y)} \end{aligned} \quad (4.20)$$

In analogy to (3.13), and because $\tilde{\xi}_3(y) < \xi_2(y)$,

$$\left. \frac{\partial \mathbb{P}[\bar{M}_1 < m, \bar{M}_3 \geq y]}{\partial y} \right|_{m=y} = - \frac{\mathbb{P}[\bar{M}_3 \geq y] - \mathbb{P}[\bar{M}_1 \geq y]}{y - \tilde{\xi}_3(y)}.$$

Hence, combining the above

$$\begin{aligned} \frac{\partial}{\partial y} \mathbb{P}[\bar{M}_3 \geq y] &= p(\tilde{\xi}_3(y), y; \tau'_2) - \frac{\mathbb{P}[\bar{M}_3 \geq y] - \mathbb{P}[\bar{M}_1 \geq y]}{y - \tilde{\xi}_3(y)} + \frac{\partial \mathbb{P}[\bar{M}_1 \geq y]}{\partial y} \\ &\stackrel{(3.10)}{=} \frac{\mathbb{P}[\bar{M}_3 \geq y]}{y - \tilde{\xi}_3(y)} - \frac{\tilde{\xi}_3(y) - \xi_1(y)}{y - \tilde{\xi}_3(y)} \frac{\partial \mathbb{P}[\bar{M}_1 \geq y]}{\partial y} + p(\tilde{\xi}_3(y), y; \tau'_2). \end{aligned} \quad (4.21)$$

In the redefined domain the first order condition for optimality of $\tilde{\xi}_3(y)$ reads

$$\tilde{K}_3(y) + c'_3(\tilde{\xi}_3(y)) - \frac{\partial f^{iAY}}{\partial \zeta}(\tilde{\xi}_3(y), y; \tau'_2) = 0. \quad (4.22)$$

By similar calculations as in (A.17) below we have

$$\begin{aligned} \tilde{K}'_3(y) &\stackrel{(4.22)}{=} - \frac{\tilde{K}_3(y)}{y - \tilde{\xi}_3(y)} - \frac{\frac{\partial f^{iAY}}{\partial y}(\tilde{\xi}_3(y), y; \tau'_2)}{y - \tilde{\xi}_3(y)} \\ &\stackrel{(4.12)}{=} - \frac{\tilde{K}_3(y)}{y - \tilde{\xi}_3(y)} + \frac{\tilde{\xi}_3(y) - \alpha(\tilde{\xi}_3(y), y)}{y - \tilde{\xi}_3(y)} \frac{\mathbb{P}[\bar{M}_1 \in dy, M_2 < \tilde{\xi}_3(y)]}{dy} \\ &\stackrel{(4.15)}{=} - \frac{\tilde{K}_3(y)}{y - \tilde{\xi}_3(y)} + \frac{\tilde{\xi}_3(y) - \xi_1(y)}{y - \tilde{\xi}_3(y)} \frac{\mathbb{P}[\bar{M}_1 \in dy]}{dy} \\ &\quad + \frac{\beta(\tilde{\xi}_3(y), y) - \tilde{\xi}_3(y)}{y - \tilde{\xi}_3(y)} \frac{\mathbb{P}[\bar{M}_1 \in dy, M_2 \geq \tilde{\xi}_3(y)]}{dy} \\ &\stackrel{(4.20)}{=} - \frac{\tilde{K}_3(y)}{y - \tilde{\xi}_3(y)} - \frac{\tilde{\xi}_3(y) - \xi_1(y)}{y - \tilde{\xi}_3(y)} \frac{\partial \mathbb{P}[\bar{M}_1 \geq y]}{\partial y} + p(\tilde{\xi}_3(y), y; \tau'_2). \end{aligned} \quad (4.23)$$

Consequently, by comparing (4.21) and (4.23), and in conjunction with Proposition 3.4, we obtain

$$\tilde{K}_3(y) = \mathbb{P}[\bar{M}_3 \geq y], \quad \text{for all } y \geq 0. \quad (4.24)$$

4.2.3 Embedding Property

After having found the distribution of the maximum, the final step is to prove the embedding property. To achieve this we will need that $\tilde{\xi}_3$ is non-decreasing.

Recall the first order condition of optimality of $\tilde{\xi}_3$ in (4.22) and then the second order condition for optimality of $\tilde{\xi}_3(y)$ reads

$$c_3''(\tilde{\xi}_3(y)) - \frac{\partial^2 f^{\text{iAY}}}{\partial \zeta^2}(\tilde{\xi}_3(y), y; \tau_2') \geq 0. \quad (4.25)$$

Now, differentiating (4.22) in y yields

$$\tilde{K}_3'(y) + c_3''(\tilde{\xi}_3(y))\tilde{\xi}_3'(y) - \frac{\partial^2 f^{\text{iAY}}}{\partial \zeta^2}(\tilde{\xi}_3(y), y; \tau_2')\tilde{\xi}_3'(y) - \frac{\partial^2 f^{\text{iAY}}}{\partial \zeta \partial y}(\tilde{\xi}_3(y), y; \tau_2') = 0$$

or equivalently,

$$\tilde{\xi}_3'(y) \underbrace{\left(c_3''(\tilde{\xi}_3(y)) - \frac{\partial^2 f^{\text{iAY}}}{\partial \zeta^2}(\tilde{\xi}_3(y), y; \tau_2') \right)}_{\geq 0 \text{ by (4.25)}} = -\tilde{K}_3'(y) + \frac{\partial^2 f^{\text{iAY}}}{\partial \zeta \partial y}(\tilde{\xi}_3(y), y; \tau_2')$$

In order to formally infer

$$\tilde{\xi}_3'(y) \geq 0$$

we require

$$-\tilde{K}_3'(y) + \frac{\partial^2 f^{\text{iAY}}}{\partial \zeta \partial y}(\tilde{\xi}_3(y), y; \tau_2') \geq 0. \quad (4.26)$$

Direct computation shows that

$$\frac{\partial^2 f^{\text{iAY}}}{\partial \zeta \partial y}(\zeta, y; \tau_2') = -\frac{\mathbb{P}[\bar{M}_1 \in dy, M_2 < \zeta]}{dy}$$

and by (4.24),

$$-\tilde{K}_3'(y) = \frac{\mathbb{P}[\bar{M}_3 \in dy]}{dy}$$

which implies (4.26) and hence that $\tilde{\xi}_3$ is non-decreasing.

By definition of the embedding in (4.7), and since $\tilde{\xi}_3$ is non-decreasing, we have

$$\begin{aligned} \mathbb{P}[\bar{M}_3 \geq y] &= \mathbb{P}[M_3 \geq \tilde{\xi}_3(y)] + \mathbb{P}[\bar{M}_3 \geq y, M_3 < \tilde{\xi}_3(y)] \\ &= \mathbb{P}[M_3 \geq \tilde{\xi}_3(y)] + \mathbb{P}[\bar{M}_1 \geq y, M_2 < \tilde{\xi}_3(y)] \\ &\stackrel{(4.11)}{=} \mathbb{P}[M_3 \geq \tilde{\xi}_3(y)] + \frac{\partial f^{\text{iAY}}}{\partial \zeta}(\tilde{\xi}_3(y), y; \tau_2'). \end{aligned} \quad (4.27)$$

and then, by (4.24), (4.22) and (4.27),

$$-c_3'(\tilde{\xi}_3(y)) = \mathbb{P}[M_3 \geq \tilde{\xi}_3(y)]$$

which is the desired embedding property.

The above construction hinged on the appropriate choice of the auxiliary term F in (4.9) whose expectation, as follows from (4.16a), (4.19) and (4.24), allows for the error book keeping, as suggested in (4.8). We identified the correct F by

analysing the “error terms” which cause strict inequality for $(B_u : u \leq \tau'_2)$ in the pathwise inequality (4.1) of Henry-Labordère et al. [10]. This is natural since this inequality is used to prove optimality of our embedding. It gives an upper bound but fails to be sharp if condition (ii) in Assumption \otimes does not hold. In order to recover a sharp bound one has to look at the error terms causing strict inequality when Assumption \otimes fails. The same principle applies for $n > 3$. However then interactions between discontinuities of boundaries $\xi_2, \tilde{\xi}_3$ etc come into play and the relevant terms become very involved. The construction would become increasingly technical and implicit and we decided to stop at this point.

A Appendix: Proof of Lemma 2.14

In order to prove Lemma 2.14 we require to prove, inductively, several auxiliary results along the way. We now state and prove a Lemma which contains the statement of Lemma 2.14.

Lemma A.1. *Let $n \in \mathbb{N}$ and let Assumption \otimes hold. Then*

$$y \mapsto K_n(y) \quad \text{is absolutely continuous and non-increasing.} \quad (\text{A.1})$$

If we assume in addition that the embedding property of Theorem 2.6 is valid for the first $n - 1$ marginals then for almost all $y \geq 0$ we have:

If $\xi_n(y) < y$ then

$$K'_n(y) + \frac{K_n(y)}{y - \xi_n(y)} = K'_{j_n(y)}(y) + \frac{K_{j_n(y)}(y)}{y - \xi_n(y)} \quad (\text{A.2})$$

where K'_j denotes the derivative of K_j which exists for almost all $y \geq 0$ and $j = 1, \dots, n$.

If $\xi_n(y) = y$ then

$$K_n(y+) = K_{j_n(y)}(y+). \quad (\text{A.3})$$

For $x > 0$ the mapping

$$c^n : (x, \infty) \rightarrow \mathbb{R}, \quad y \mapsto c^n(x, y) \quad (\text{A.4})$$

is locally Lipschitz continuous, non-decreasing and for almost all $y > 0$

$$\left. \frac{\partial c^n}{\partial y}(x, y) \right|_{x=\xi_n(y)} = K_{j_n(y)}(y) + (y - \xi_n(y))K'_{j_n(y)}(y). \quad (\text{A.5})$$

The mapping $c^n(\cdot, y)$ is locally Lipschitz continuous and if it is differentiable at $\xi_n(y)$ and $\xi'_n(y) > 0$ then for almost all $y \geq 0$

$$K_n(y) + c'_n(\xi_n(y)) - c'_j(\xi_n(y)) - K_j(y) = 0 \quad (\text{A.6})$$

for $j = j_n(y)$ and j such that $n > j > j_n(y)$ and $\xi_n(y) = \xi_j(y)$. In the case of non-smoothness of $c^n(\cdot, y)$ at $\xi_n(y)$ we have

$$\xi'_n(y) = 0 \quad (\text{A.7})$$

for y such that the slope of the supporting tangent to $c^n(\cdot, y)$ at $\xi_n(y)$ which crosses the x -axis at y does not equal the right-derivative of $c^n(\cdot, y)$ at $\xi_n(y)$.

Proof. We prove the claim by induction over n . The induction basis $n = 1$ holds by definition and Lemma 2.6 of Brown et al. [4].

Now assume that the claim holds for all $i = 1, \dots, n - 1$.

Induction step for c^n . We have

$$\begin{aligned} c^n(x, y + \delta) - c^n(x, y) &= - [c_{\iota_n(x; y + \delta)}(x) - (y + \delta - x)K_{\iota_n(x; y + \delta)}(y + \delta)] \\ &\quad + [c_{\iota_n(x; y)}(x) - (y - x)K_{\iota_n(x; y)}(y)]. \end{aligned} \quad (\text{A.8})$$

Firstly, consider the case when there exists a $\delta' > 0$ such that for all $|\delta| < \delta'$ we have $\iota_n(x; y) = \iota_n(x; y + \delta)$. Equation (A.8) simplifies and we have

$$\begin{aligned} c^n(x, y + \delta) - c^n(x, y) &= (y + \delta - x)K_{\iota_n(x; y)}(y + \delta) - (y - x)K_{\iota_n(x; y)}(y) \\ &= (y + \delta - \xi_{\iota_n(x; y)}(y))K_{\iota_n(x; y)}(y + \delta) - (y - \xi_{\iota_n(x; y)}(y))K_{\iota_n(x; y)}(y) \\ &\quad + (x - \xi_{\iota_n(x; y)}(y)) [K_{\iota_n(x; y)}(y) - K_{\iota_n(x; y)}(y + \delta)] \\ &\stackrel{(2.14)}{\leq} c^{\iota_n(x; y)}(\xi_{\iota_n(x; y)}(y), y + \delta) - c^{\iota_n(x; y)}(\xi_{\iota_n(x; y)}(y), y) \\ &\quad + (y - \xi_{\iota_n(x; y)}(y)) [K_{\iota_n(x; y)}(y) - K_{\iota_n(x; y)}(y + \delta)] \\ &\leq \max_{i < n} \left\{ c^i(\xi_i(y), y + \delta) - c^i(\xi_i(y), y) + (y - \xi_i(y)) [K_i(y) - K_i(y + \delta)] \right\} \\ &\leq \text{const}(y) \cdot |\delta| \end{aligned} \quad (\text{A.9})$$

by induction hypothesis and where $\text{const}(y)$ denotes a constant depending on y . A similar computation shows for $|\delta|$ small enough

$$\begin{aligned} c^n(x, y) - c^n(x, y + \delta) &\stackrel{(2.14)}{\leq} c^{\iota_n(x; y)}(\xi_{\iota_n(x; y)}(y + \delta), y) - c^{\iota_n(x; y)}(\xi_{\iota_n(x; y)}(y + \delta), y + \delta) \\ &\quad + (x - \xi_{\iota_n(x; y)}(y + \delta)) [K_{\iota_n(x; y)}(y + \delta) - K_{\iota_n(x; y)}(y)] \\ &= (y - \xi_{\iota_n(x; y)}(y + \delta)) [K_{\iota_n(x; y)}(x; y)(y) - K_{\iota_n(x; y)}(x; y)(y + \delta)] - \delta K_{\iota_n(x; y)}(x; y)(y + \delta) \\ &\quad + (x - \xi_{\iota_n(x; y)}(y + \delta)) [K_{\iota_n(x; y)}(y + \delta) - K_{\iota_n(x; y)}(y)] \\ &\leq \begin{cases} \text{const}(y) \cdot |\delta| & \text{if } \delta < 0, \\ 0 & \text{if } \delta \geq 0, \end{cases} \end{aligned} \quad (\text{A.10})$$

again by induction hypothesis and continuity of ξ_i which indeed allows the constant to be chosen independently of δ . Monotonicity of $c^n(x, \cdot)$ follows. Equation (A.10) together with (A.9) imply the local Lipschitz continuity. Plugging $x = \xi_n(y)$ into (A.8), direct computation shows that (A.5) holds.

Secondly, consider the case when $\iota_n(x; \cdot)$ jumps at y . Recall (2.9). Note that this is only possible when x satisfies

$$x = \xi_{\iota_n(x; y - \delta)}(y) \quad \text{for } \delta > 0 \text{ small enough,} \quad (\text{A.11})$$

i.e. when $x = \xi_k(y)$ for the index $k = \iota_n(x; y - \delta) > j_n(y)$. By (2.9) there exists a $\delta' > 0$ such that $\iota_n(x; y + \delta) = \iota_n(x; y)$ for all $0 \leq \delta < \delta'$. Hence, for $\delta > 0$ small enough, $|c^n(x, y + \delta) - c^n(x, y)|$ has the same upper bound as in the first case. Monotonicity of $c^n(x, \cdot)$ follows.

Furthermore, for $\delta > 0$ we have for the x from (A.11) that $\iota_n(x; y - \delta) > \iota_n(x; y)$ holds. For notational simplicity we only consider the case

$$\iota_{\iota_n(x; y - \delta)}(x; y - \delta) = \iota_n(x; y).$$

The general case follows by the same arguments. We deduce from (A.8) the following two equations,

$$\begin{aligned} c^n(x, y - \delta) - c^n(x, y) &\stackrel{(2.14)}{\leq} (y - \delta - x)K_{\iota_n(x; y - \delta)(x; y - \delta)}(y - \delta) - (y - x)K_{\iota_n(x; y)}(y) \\ &= (y - \delta - x)K_{\iota_n(x; y)}(y - \delta) - (y - x)K_{\iota_n(x; y)}(y) \end{aligned} \quad (\text{A.12})$$

and

$$c^n(x, y) - c^n(x, y - \delta) \stackrel{(\text{A.11})}{=} -(y - \delta - x)K_{\iota_n(x; y - \delta)}(y - \delta) + (y - x)K_{\iota_n(x; y - \delta)}(y).$$

Now the local Lipschitz continuity of $c^n(x, \cdot)$ follow from the above two equations by repeating the arguments from the first case.

We prove (A.5) by computing the required right- and left-derivative of $c^n(x, \cdot)$ at $x = \xi_n(y)$. The right-derivative is simply, using (2.9) and (A.8),

$$K_{J_n(y)}(y) + (y - \xi_n(y))K'_{J_n(y)}(y) \quad (\text{A.13})$$

and the left-derivative is, writing $k = \iota_n(\xi_n(y); y-) > \iota_n(\xi_n(y); y) = J_n(y)$,

$$\begin{aligned} &\lim_{\delta \uparrow 0} \frac{1}{\delta} \left(-c_k(\xi_n(y)) + (y + \delta - \xi_n(y))K_k(y + \delta) \right. \\ &\quad \left. + c_{J_n(y)}(\xi_n(y)) - (y - \xi_n(y))K_{J_n(y)}(y) \right) \\ \stackrel{\xi_n(y) = \xi_k(y)}{=} &\lim_{\delta \uparrow 0} \frac{1}{\delta} \left((y + \delta - \xi_n(y))K_k(y + \delta) - (y - \xi_n(y))K_k(y) \right) \\ = &K_k(y) + (y - \xi_n(y))K'_k(y) \stackrel{(\text{A.2})}{=} K_{J_n(y)}(y) + (y - \xi_n(y))K'_{J_n(y)}(y) \end{aligned} \quad (\text{A.14})$$

by induction hypothesis. So the two coincide for almost all $y > 0$.

Induction step for K_n . A straightforward computation shows that the mapping $y \mapsto \frac{c^n(x, y)}{y - x}$ is non-increasing and hence for $\delta > 0$

$$K_n(y + \delta) = \inf_{\zeta \leq y + \delta} \frac{c^n(\zeta, y + \delta)}{y + \delta - \zeta} \leq \inf_{\zeta \leq y} \frac{c^n(\zeta, y + \delta)}{y + \delta - \zeta} \leq \inf_{\zeta \leq y} \frac{c^n(\zeta, y)}{y - \zeta} = K_n(y)$$

proving that K_n is non-increasing.

Using again that $c^n(x, \cdot)$ is non-decreasing and that ξ_n is continuous, local Lipschitz continuity of K_n now follows from

$$K_n(y) \leq \frac{c^n(\xi_n(y + \delta), y)}{y - \xi_n(y + \delta)} \leq \frac{c^n(\xi_n(y + \delta), y + \delta)}{y - \xi_n(y + \delta)} = K_n(y + \delta) \left(1 + \frac{\delta}{y - \xi_n(y + \delta)} \right)$$

if $\xi_n(y) < y$ and if $\xi_n(y) = y$, recalling (2.15), we have

$$\begin{aligned} K_n(y+) &= \inf_{\zeta \leq (y+)} \left\{ \frac{c_n(\zeta) - c_{\iota_n(\zeta; y+)}(\zeta)}{(y+) - \zeta} + K_{\iota_n(\zeta; y+)}(y) \right\} \\ &\stackrel{\text{Lemma 2.13}}{=} \inf_{y \leq \zeta \leq (y+)} \left\{ \frac{c_n(\zeta) - c_{\iota_n(\zeta; y+)}(\zeta)}{(y+) - \zeta} + K_{\iota_n(\zeta; y+)}(y+) \right\} = K_{\iota_n(y; y)}(y+) \\ &= K_{J_n(y)}(y+), \end{aligned}$$

and local Lipschitz continuity of K_n follows by induction hypothesis. Equation (A.3) is also proven.

Local Lipschitz continuity of $c^n(\cdot, y)$ follows from the properties of ι_n , cf. (2.8), the fact that the functions $c_i, i = 1, \dots, n$, are locally Lipschitz and a similar expansion of terms in the case when $\xi_n(y) = \xi_i(y)$ for some $i < n$

In order to prove (A.6) we first exclude all $y \geq 0$ such that $\xi_n(y)$ is an atom of c_1, \dots, c_n and $\xi'_n(y) > 0$. Amongst all $y \in \{\xi'_n > 0\}$ this is a null-set. By assumption $c^n(\cdot, y)$ is differentiable at $\xi_n(y)$. Recalling the equations (2.35)–(2.37), a direct computation proves (A.6) for $k = \nu_n(\xi_n(y); y)$. Now we want to apply the induction hypothesis to c^k . By choice of y we have that c_k is differentiable at $\xi_n(y) = \xi_k(y)$, i.e. μ_k does not have an atom at $\xi_n(y)$. Hence, by the assumption that the embedding for the first $n - 1$ marginals is valid we cannot have $\xi'_k(y) = 0$ (except on a null-set because otherwise the embedding would fail). By (A.7), $c^k(\cdot, y)$ therefore has to be differentiable at $\xi_n(y) = \xi_k(y)$. This shows that we can indeed apply (A.6) to deduce for $j = j_n(y)$ and j such that $n > j > j_n(y)$ and $\xi_n(y) = \xi_k(y) = \xi_j(y)$ the following equation,

$$\begin{aligned} 0 &= K_n(y) + c'_n(\xi_n(y)) - c'_k(\xi_n(y)) + K_k(y) \\ &= K_n(y) + c'_n(\xi_n(y)) - c'_k(\xi_n(y)) + c'_k(\xi_n(y)) - c'_j(\xi_n(y)) + K_j(y) \\ &= K_n(y) + c'_n(\xi_n(y)) - c'_j(\xi_n(y)) + K_j(y). \end{aligned}$$

Equation (A.6) is proven.

For later use we note the equation

$$\frac{c_n(\xi_n(y)) - c_{j_n(y)}(\xi_n(y))}{y - \xi_n(y)} + K_{j_n(y)}(y) = K_n(y) = \frac{c_n(\xi_n(y)) - c_k(\xi_n(y))}{y - \xi_n(y)} + K_k(y) \quad (\text{A.15})$$

for k such that $n > k > j_n(y)$ and $\xi_n(y) = \xi_k(y)$.

Finally, we prove the claimed ODE for K_n in the case $\xi_n(y) < y$. For almost all $y \geq 0$ we have

$$\begin{aligned} K'_n(y) &= \lim_{\delta \rightarrow 0} \frac{1}{\delta} \left[\frac{c^n(\xi_n(y + \delta), y + \delta)}{y + \delta - \xi_n(y + \delta)} - \frac{c^n(\xi_n(y), y)}{y - \xi_n(y)} \right] \\ &= \lim_{\delta \rightarrow 0} \frac{1}{\delta} \left[\left(\frac{1}{y + \delta - \xi_n(y + \delta)} - \frac{1}{y - \xi_n(y)} \right) c^n(\xi_n(y + \delta), y + \delta) \right. \\ &\quad \left. + \frac{c^n(\xi_n(y + \delta), y + \delta) - c^n(\xi_n(y), y)}{y - \xi_n(y)} \right] \\ &= \frac{\xi'_n(y) - 1}{y - \xi_n(y)} K_n(y) + \frac{1}{y - \xi_n(y)} \left(\lim_{\delta \rightarrow 0} \frac{c^n(\xi_n(y + \delta), y + \delta) - c^n(\xi_n(y), y)}{\delta} \right). \end{aligned}$$

The main technical difficulty comes from the possibility that $\xi_n(y) = \xi_k(y)$ for some $k < n$. We present the arguments for this case and leave the other (much easier) case, to the reader.

By assumption the last limit exists and hence we can compute it using some “convenient” sequence $\delta_m \downarrow 0$ where δ_m is such that $j_n(y + \delta_m) = l$ for all $m \in \mathbb{N}$. Note that by continuity of ξ_1, \dots, ξ_n at y we have that either $l = j_n(y)$ or l is such that $\xi_l(y) = \xi_n(y)$. This will enable us apply (A.15). Recall (2.9). For δ_m small enough such that $\nu_n(\xi_n(y); y + \delta_m) = j_n(y)$ we obtain

$$\begin{aligned} &c^n(\xi_n(y + \delta_m), y + \delta_m) - c^n(\xi_n(y), y + \delta_m) \\ &= c_n(\xi_n(y + \delta_m)) - c_l(\xi_n(y + \delta_m)) + (y + \delta_m - \xi_n(y + \delta_m))K_l(y + \delta_m) \\ &\quad - c_n(\xi_n(y)) + c_{j_n(y)}(\xi_n(y)) - (y + \delta_m - \xi_n(y))K_{j_n(y)}(y + \delta_m) \\ &\stackrel{(\text{A.15})}{=} c_n(\xi_n(y + \delta_m)) - c_l(\xi_n(y + \delta_m)) + (y + \delta_m - \xi_n(y + \delta_m))K_l(y + \delta_m) \\ &\quad - c_n(\xi_n(y)) + c_l(\xi_n(y)) - (y - \xi_n(y))(K_l(y) - K_{j_n(y)}(y)) \\ &\quad - (y + \delta_m - \xi_n(y))K_{j_n(y)}(y + \delta_m). \end{aligned}$$

From this we obtain for almost all $y \geq 0$ by using the induction hypothesis

$$\begin{aligned}
& \lim_{m \rightarrow \infty} \frac{c^n(\xi_n(y + \delta_m), y + \delta_m) - c^n(\xi_n(y), y + \delta_m)}{\delta_m} \\
&= \xi'_n(y+) \left[c'_n(\xi_n(y)+) - c'_l(\xi_n(y)+) - K_l(y) \right] \\
&\quad + K_l(y) + (y - \xi_n(y))K'_l(y) - K_{j_n(y)}(y) - (y - \xi_n(y))K'_{j_n(y)}(y) \\
&\stackrel{(A.6)}{=} - \xi'_n(y+)K_n(y). \tag{A.16}
\end{aligned}$$

Together with (A.5) this yields in the case when $c^n(\cdot, y)$ is differentiable at $\xi_n(y)$

$$\begin{aligned}
K'_n(y) &= \frac{\xi'_n(y) - 1}{y - \xi_n(y)} K_n(y) + \frac{1}{y - \xi_n(y)} \left(-K_n(y)\xi'_n(y) + \frac{\partial c^n}{\partial y}(\xi_n(y), y) \right) \\
&= -\frac{K_n(y)}{y - \xi_n(y)} + \frac{1}{y - \xi_n(y)} \left(K_{j_n(y)}(y) + (y - \xi_n(y))K'_{j_n(y)}(y) \right). \tag{A.17}
\end{aligned}$$

In order to finish the proof we just have to establish that (A.17) also holds in the case when $c^n(\cdot, y)$ is not differentiable at $\xi_n(y)$.

To this end, we first argue that (A.16), and hence (A.17), remains true in the case when $c^n(\cdot, y)$ is not differentiable at $\xi_n(y)$, but when the slope of the supporting tangent to $c^n(\cdot, y)$ at $\xi_n(y)$ which passes the x -axis at y equals the right-derivative of $c^n(\cdot, y)$ at $\xi_n(y)$. In that case, denoting $k = j_n(\xi_n(y)+; y)$,

$$c'_n(\xi_n(y)+) - c'_k(\xi_n(y)+) - K'_k(y) = -K_n(y). \tag{A.18}$$

Recall the sequence (δ_m) . We do not necessarily have $k = j_n(y + \delta_m) = l$. Nevertheless, we argue that

$$c'_n(\xi_n(y)+) - c'_l(\xi_n(y)+) - K_l(y) = -K_n(y) \tag{A.19}$$

holds. We can safely assume that $\xi'_n(y+) > 0$ (in the other case the conclusion of (A.17) remains true). Also it is enough to consider the case when $k > j_n(y + \delta)$ for all $\delta > 0$ sufficiently small (otherwise we may consider an alternative sequence (δ_m) where $k = j_n(y + \delta_m)$ for all m and (A.19) would follow from (A.18)). Consequently, $\xi'_k(y+) \geq \xi'_n(y+) > 0$. Then, since by induction hypothesis (A.6) holds true for k , we must have, for almost all y that

$$c'_k(\xi_n(y)+) - c'_j(\xi_n(y)+) - K'_j(y) = -K_k(y) \tag{A.20}$$

Then, combining these results we conclude by (A.18) and (A.20) that indeed (A.19) holds.

Now we consider the case of non-smoothness of $c^n(\cdot, y)$ at $\xi_n(y)$ and where y is such that the slope of the supporting tangent to $c^n(\cdot, y)$ at $\xi_n(y)$ which crosses the x -axis at y does not equal the right-derivative of $c^n(\cdot, y)$ at $\xi_n(y)$. We show that in this case we have for sufficiently small $\delta > 0$,

$$\xi_n(y) = \xi_n(y + \delta) \quad \text{and hence} \quad \xi'_n(y+) = 0, \tag{A.21}$$

which implies that (A.17) holds as well.

To achieve this we place a suitable tangent to $c^n(\cdot, y)$ at $\xi_n(y)$. Since, by assumption, $c^n(\cdot, y)$ has a kink at $\xi_n(y)$ we have some flexibility to do that. Recalling the tangent interpretation of (2.14) we know by choice of $\xi_n(y)$ that we can place a supporting tangent to $c^n(\cdot, y)$ at $\xi_n(y)$ which passes through the x -axis at y . Alternatively, by choice of y , we can place a tangent to $c^n(\cdot, y)$ at $\xi_n(y)$ which crosses the x -axis at some $y + \delta > y$. This implies that

$$\arg \min_{\zeta \leq y} \frac{c^n(\zeta, y)}{y + \delta - \zeta} = \xi_n(y). \tag{A.22}$$

Assume first that $\xi_n(y) < y$. Denote $k = \iota_n(\xi_n(y); y)$. For simplicity of the argument let us also assume that $\xi_n(y) = \xi_k(y) \neq \xi_i(y)$ for all $i \neq k, n$. Also denote $j = \jmath_n(y) = \iota_n(\xi_n(y); y)$.

Now we will use (A.22) to deduce (A.21). By continuity and monotonicity of ξ_n we have for $\delta > 0$ small enough that $\xi_n(y) \leq \xi_n(y + \delta) < \xi_n(y) + \epsilon < y$ for some $\epsilon = \epsilon(\delta) > 0$. By taking δ small enough we can also assume that $k = \max_{\zeta \leq \xi_n(y) + \epsilon} \iota_n(\zeta; y + \delta)$. Then we have

$$\inf_{\zeta \leq y + \delta} \frac{c^n(\zeta, y + \delta)}{y + \delta - \zeta} \geq \inf_{\xi_n(y) \leq \zeta < \xi_n(y) + \epsilon} \frac{c^n(\zeta, y)}{y + \delta - \zeta} + \inf_{\xi_n(y) \leq \zeta \leq \xi_n(y) + \epsilon} \frac{c^n(\zeta, y + \delta) - c^n(\zeta, y)}{y + \delta - \zeta} \quad (\text{A.23})$$

As for the first infimum in (A.23) we know from (A.22) that it is attained at $\zeta = \xi_n(y)$. Now we will show that the second infimum in (A.23) is also attained at $\zeta = \xi_n(y)$. To this end consider the following estimate for $\zeta > \xi_n(y)$,

$$\begin{aligned} & c^n(\zeta, y + \delta) - c^n(\zeta, y) \\ &= -c_{\iota_n(\zeta; y + \delta)}(\zeta) + c_{\iota_n(\zeta; y)}(\zeta) - (y - \zeta)K_{\iota_n(\zeta; y)}(y) + (y + \delta - \zeta)K_{\iota_n(\zeta; y + \delta)}(y + \delta) \\ &\stackrel{(2.14)}{\geq} - (y - \zeta)K_{\iota_n(\zeta; y)}(\zeta; y) + (y + \delta - \zeta)K_{\iota_n(\zeta; y + \delta)}(y + \delta) \\ &\geq - (y - \zeta)K_j(y) + (y + \delta - \zeta)K_j(y + \delta) =: l(\zeta, y; \delta). \end{aligned} \quad (\text{A.24})$$

Since $l(\cdot, y; \delta)$ is non-decreasing and non-negative, we deduce that

$$\arg \min_{\xi_n(y) \leq \zeta < \xi_n(y) + \epsilon} \frac{l(\zeta, y; \delta)}{y + \delta - \zeta} = \xi_n(y).$$

Finally, because at $\zeta = \xi_n(y)$ there is equality in (A.24) we can conclude

$$\arg \min_{\zeta \leq y + \delta} \frac{c^n(\zeta, y + \delta)}{y + \delta - \zeta} = \xi_n(y)$$

as required.

In the case when $\xi_n(y) = y$ we obtain by (2.9) for $\delta > 0$ sufficiently small that $\iota_n(y; y + \delta) = \iota_n(y; y)$ and hence

$$\begin{aligned} \arg \min_{\zeta < y + \delta} \frac{c^n(\zeta, y + \delta)}{y + \delta - \zeta} &= \arg \min_{y \leq \zeta < y + \delta} \left(\underbrace{\frac{c_n(\zeta) - c_{\iota_n(\zeta; y + \delta)}(\zeta)}{y + \delta - \zeta}}_{\geq 0} + \underbrace{K_{\iota_n(\zeta; y + \delta)}(y + \delta)}_{\geq K_{\iota_n(y; y + \delta)}(y + \delta)} \right) \\ &\stackrel{(2.15)}{=} \xi_n(y) = y. \end{aligned}$$

The proof is complete. \square

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