Commodity Storage Valuation

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Oxford-Man Institute
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Storing commodities

- Non-perishable commodities are stored in a range of facilities. Eg.
  - Tanks for oil
  - Underground caverns for natural gas
  - Reservoirs and dams for water
  - Silos for agricultural commodities
- The value of a facility would be the expected discounted value of optimally using the storage and trading the commodity.

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- The value of a facility would be the expected discounted value of optimally using the storage and trading the commodity.

- We focus on the determination of optimal injection/withdrawal policies and the valuation of storage facilities.
  - Offer a unified pricing framework for the case of a small (price-taker) storage facility and
  - A computational algorithm to solve for the value and the optimal policy.

Model Formulation

- Commodity price
  - Price process S_t modeled by (Schwartz 1997)

$$dS_t = \kappa(\gamma - \ln S_t)S_t dt + \sigma S_t dZ_t$$

  - With $X_t = \ln S_t$, we have the Ornstein-Uhlenbeck process

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dZ_t$$

where $\alpha = \gamma - \sigma^2/(2\kappa)$.

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- Storage level
  - $Q_t \in [0, Q_{\max}]$,
  - Can be adjusted (controlled) by buying or selling at $S_t \equiv e^{X_t}$
  - Will also have costs of control.

Cost of changing stored quantity

- Per unit cost of injection, $\lambda(q)$, is non-decreasing and convex
- Hence when price is e^x the cost of getting the stored quantity from Q to η would be

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- Under such costs we have a singular control problem and the dynamics of Q can be written as

$$dQ_t = dL_t - dU_t$$

(L_t, U_t) represent cumulative injections, withdrawals.

Objective

- Objective: to maximize discounted infinite-horizon discounted cash flows.

$$V(x,q) = \max E_{x,q} \left[\int_0^\infty e^{-\beta t}(e^{X_t} - \mu(Q_t))dU_t - \int_0^\infty e^{-\beta t}(e^{X_t} + \lambda(Q_t))dL_t \right]$$

where, discount factor $\beta \in (0,1)$ and $X_0 = x$ and $Q_0 = q$.

The Hamilton Jacobi Bellman Equation

- Dynamic programming arguments and Ito's formula yield the Hamilton-Jacobi-Bellman (HJB) equation.

$$\max\left(\mathcal{L}V,\ \ \frac{\partial V}{\partial q} - (e^x + \lambda(q)),\ \ -\frac{\partial V}{\partial q} + (e^x - \mu(q))\right) = 0$$

with $\mathcal{L}V \equiv \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} + \alpha(\kappa - x)\frac{\partial V}{\partial x} - \beta V$

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- The state space $(x, q) \in \mathcal{R} \times [0, Q_{\max}]$ can have three kinds of regions
  - injection region: $\frac{\partial V}{\partial q} - (e^x + \lambda(q)) = 0$,
  - withdrawal region: $-\frac{\partial V}{\partial q} + (e^x - \mu(q)) = 0$ and
  - a hold region: $\mathcal{L}V = 0$.

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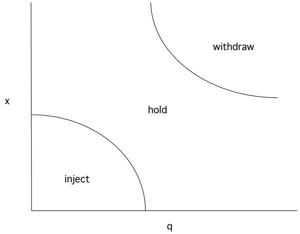
  - a hold region: $\mathcal{L}V = 0$.
- A verification theorem assures us that a function that solves the HJB equation is the value function for the original control problem and a policy that achieves this value function is the optimal policy.

HJB equation graphically

- Find $V(x, q)$, and the three regions, Ω, Ω_i and Ω_w such that
  - $\mathcal{L}V = 0$ in Ω,
  - $\frac{\partial V}{\partial q} - (e^x + \lambda(q)) = 0$ in Ω_i,
  - $-\frac{\partial V}{\partial q} + (e^x - \mu(q)) = 0$ in Ω_w and
  - $\max\{\mathcal{L}V, \frac{\partial V}{\partial q} - (e^x + \lambda(q)), -\frac{\partial V}{\partial q} + (e^x - \mu(q)) = 0\}$ for all (x, q).



- No reason yet to believe the regions are connected...

Structure of optimal boundaries

- Can show the following
  - If it is optimal to inject at some state (x, q), then it is optimal to inject at any state $(x, \bar{q})$ for $\bar{q} \in [0, q]$.
  - If it is optimal to withdraw at some state (x, q), then it is optimal to withdraw at any state $(x, \bar{q})$ for $\bar{q} \in [q, Q_{\max}]$.

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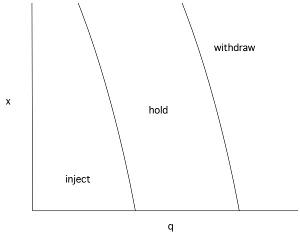

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  - as $x \to \infty$ optimal to always empty the storage
  - as $x \to -\infty$ optimal to hold

Structure of optimal boundaries

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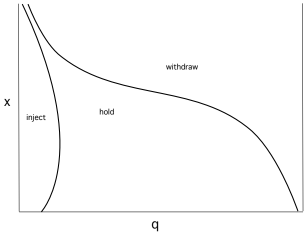
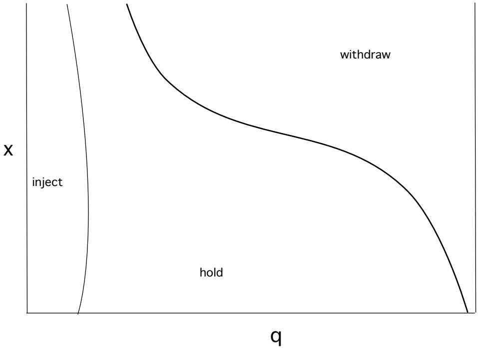
  - as $x \to \infty$ optimal to always empty the storage
  - as $x \to -\infty$ optimal to hold
- these imply that $i(x) = w(x) = 0$ as $x \to \infty$ and $i(x) = 0, w(x) = Q_{\max}$ as $x \to -\infty$.



The Moving Boundary Method



- Say Ω^0 is such that $\Omega^* \subset \Omega^0$: The HJB would be violated.

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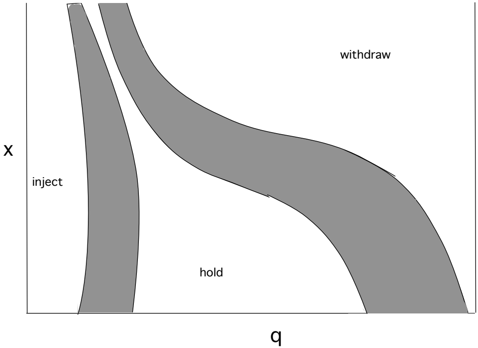

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- Any Ω^1 with bds in the shaded region: Policy improvement ($\mathbf{V^1} > \mathbf{V^0}$), but possibly $\Omega^* \not\subset \Omega^1$.

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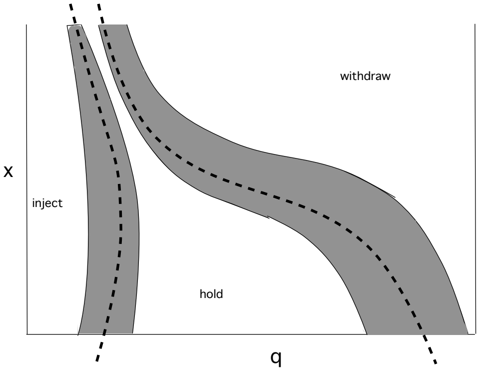

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- Any Ω^1 with bds in the shaded region: Policy improvement ($\mathbf{V^1} > \mathbf{V^0}$), but possibly $\Omega^* \not\subset \Omega^1$.
- If bds are chosen carefully then we can assure policy improvement and that $\Omega^* \subset \Omega^1$.

The update equations

- The boundary update equations

$$
\begin{aligned}
i^{n+1}(x) &= \sup\left\{\bar{q} \in (i^n(x), w^n(x)) \Big| \frac{\partial^2 V^n}{\partial q^2} > \frac{\partial \lambda}{\partial q} \ \ \forall q < \bar{q}\right\} \qquad \text{and} \\
w^{n+1}(x) &= \inf\left\{\bar{q} \in (i^n(x), w^n(x)) | \frac{\partial^2 V^n}{\partial q^2} > \frac{\partial \mu}{\partial q} \ \ \forall q > \bar{q}\right\}
\end{aligned}
$$

- These are essentially the equations to draw the new boundaries as the conturs of points of maximum violation of the HJB.

The Moving Boundary Method

- Start with $i^0(x) = 0$ and $w^0(x) = Q_{\max}$
- Solve for V^n using $\mathcal{L}V^n = 0$ in (i^n, w^n)
- Update i^n, w^n to i^{n+1}, w^{n+1} and iterate until convergence

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Theoretical Guarantees:

- The sequence of boundaries are monotone and improving.
- If the converged function is sufficiently smooth then it is the optimal value function.

Solving the fixed boundary problem

- In the hold region

$$\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + \alpha(\kappa - x)\frac{\partial V}{\partial x} - \beta V = 0$$

- defining $y = \kappa(x-\alpha)^2/\sigma^2$, we have

$$y\frac{\partial^2 V}{\partial y^2} + (0.5 - y)\frac{\partial V}{\partial y} - \frac{\beta}{2\kappa}V = 0$$

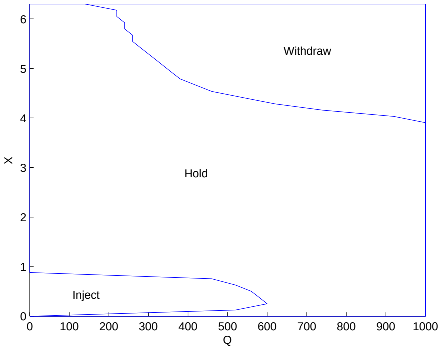
which is the Kummer Equation. The solution for which is known to be the sum of hypergeometric1F1 and the hypergeometricU functions. Hence,

$$\begin{aligned} V(x,q) &= A(q) HyperGeoU\left(\frac{\beta}{2\kappa}, \frac{1}{2}, \frac{\kappa}{\sigma^2}(x-\alpha)^2\right) + \\ &\quad B(q) HyperGeo1F1\left(\frac{\beta}{2\kappa}, \frac{1}{2}, \frac{\kappa}{\sigma^2}(x-\alpha)^2\right) \end{aligned}$$

Boundary conditions provide eqns to determine $A(q)$ and $B(q)$.

An Oil Storage Example

- Using data from 1985 to 1995: $\sigma = 0.334, \kappa = 0.0301$ and $\alpha = 3.15$.
- With discount rate $\beta = 5\%$ per year and
- $Q_{\max} = 1000, \lambda(q) = \frac{1000}{1000-q} - 1$ and $\mu(q) = \frac{1000}{q} - 1$.



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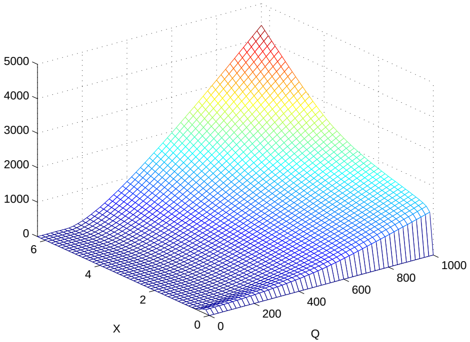

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Cost of changing stored quantity - fixed

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- Such costs make infinitesimal changes sub-optimal and require modeling via Impulse controls which make non-infinitesimal changes
- Control is modeled by a sequence of times τ_i and a sequence of non-zero increments ξ_i.
- $Q(\tau_i) = Q(\tau_i-) + \xi_i$ for $i = 1, 2, \ldots$

The HJB

- HJB now becomes

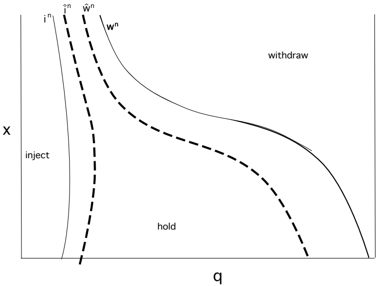
$$\max\left(\mathcal{L}V, \mathcal{I}V, \mathcal{W}V\right) = 0$$

where
$\mathcal{I}V \equiv \max_{\eta\in(q,Q_{max}]} V(\eta) - V(q) - K_i - \int_q^\eta \lambda(q)dq - e^x(\eta - q)$
and $\mathcal{W}V \equiv \max_{\eta\in[0,q)} V(\eta) - V(q) - K_w - \int_\eta^q \mu(q)dq - e^x(q - \eta)$

- The state space can again have three kinds of regions
  - injection region: $\mathcal{I}V = 0$,
  - withdrawal region: $\mathcal{W}V = 0$ and
  - a hold region: $\mathcal{L}V = 0$.

Regions

- The state space can still be divided into three kinds of regions
- however we will have a '*jump from*' as well as a '*jump to*' boundary
  - injection region: $\mathcal{I}V = 0$,
  - withdrawal region: $\mathcal{W}V = 0$ and
  - a hold region: $\mathcal{L}V = 0$.



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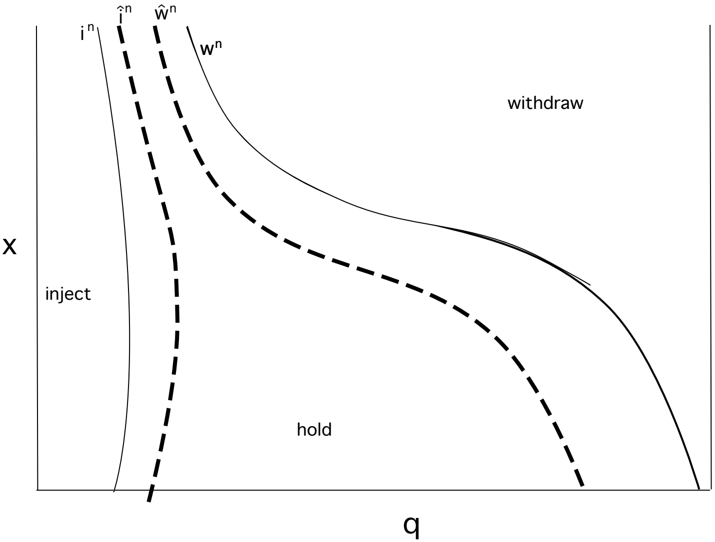

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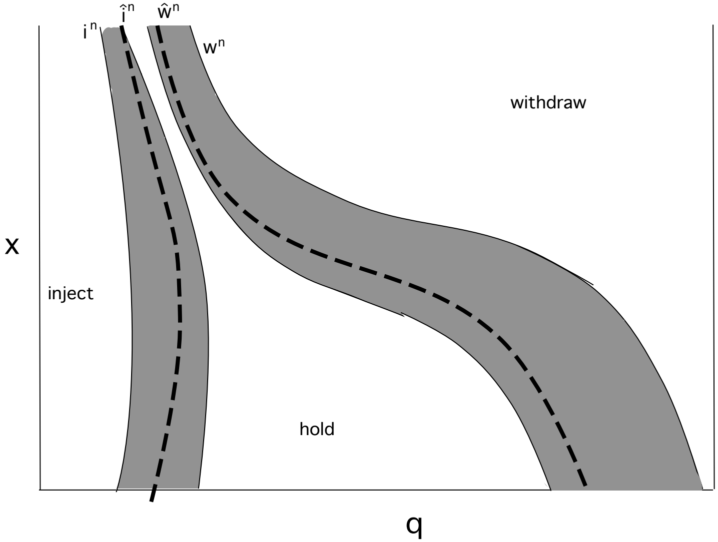

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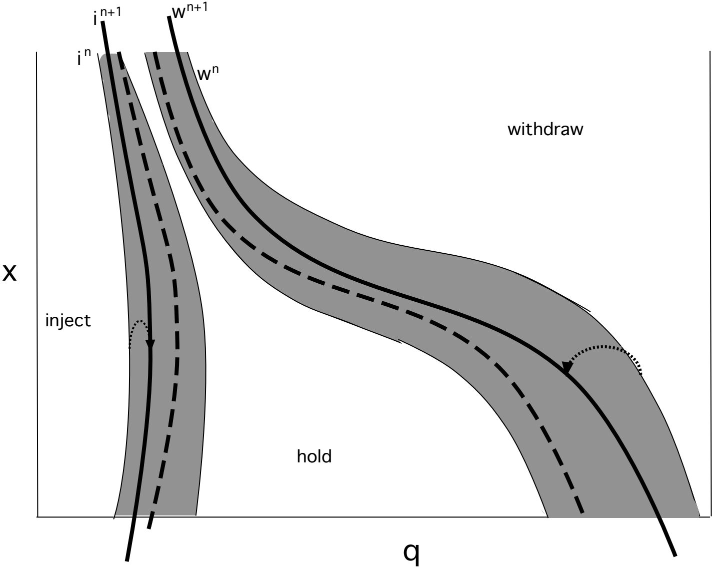

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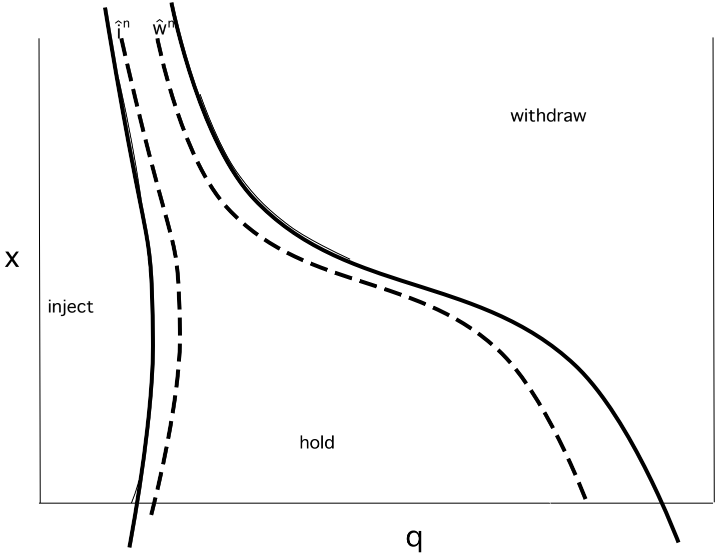

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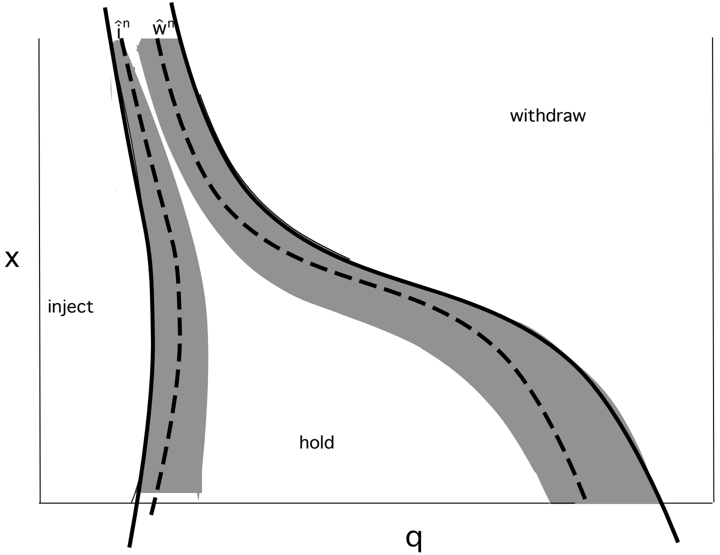

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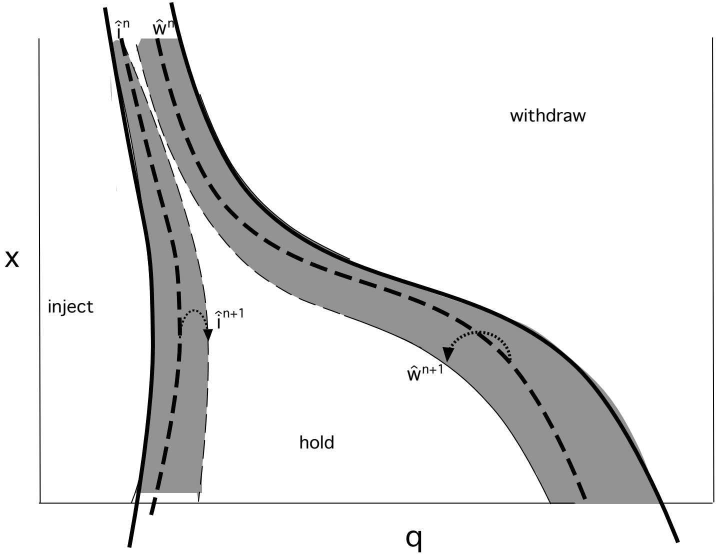

The Moving Boundary Method

- Solve for V using $(i^n, \hat{i}^n, \hat{w}^n, w^n)$
- Update boundaries i, w

$$
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i^{n+1}(x) &= \sup\left\{\bar{q} \in (i^n(x), \hat{i}^n(x)) \middle| \frac{\partial V^n}{\partial q} < e^x + \lambda \ \ \forall q < \bar{q}\right\} \quad \text{and} \\
w^{n+1}(x) &= \inf\left\{\bar{q} \in (\hat{w}^n(x), w^n(x)) \middle| \frac{\partial V^n}{\partial q} > e^x - \mu \ \ \forall q > \bar{q}\right\}
\end{aligned}
$$

- Next solve for $\hat{V}$ using $(i^{n+1}, \hat{i}^n, \hat{w}^n, w^{n+1})$
- Update boundaries $\hat{i}, \hat{w}$ using

$$
\begin{aligned}
\hat{i}^{n+1}(x) &= \text{argsup}_\eta \left\{\hat{V}(\eta) - \int_{i^{n+1}(x)}^{\eta} \lambda(q)dq - e^x\eta\right\} \quad \text{and} \\
\hat{w}^{n+1}(x) &= \text{argsup}_\eta \left\{\hat{V}(\eta) - \int_{\eta}^{w^{n+1}(x)} \mu(q)dq - e^x\eta\right\}
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- This paper essentially uses the commodity valuation application to extend the moving boundary method to OU processes
  - thereby extending it to work for folding boundaries,
  - provides proof of convergence of the method is a setting with more than 1-dimension (for the first time)
  - and the fixed boundary problems can be solved analytically (for the first time)

Questions, Comments?