Extrapolation methods for weak approximation schemes

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Basic setting

- underlying modelled by a stochastic process $(X_t)_{t \geq 0}$

$$dX_t = AX_t\,dt + \sum_{i=0}^{d} V_i(X_t) \circ dB_t^i, \quad X_0 = x,$$

  - state space: separable Hilbert space $(H, \|.\|_H)$
  - vector fields V_i, $i = 0, \ldots, d$ Lipschitz
  - A: unbounded operator on H
  - A generates C^0 pseudocontractible semigroup
  - $B_t^0 = t$, $(B_t^i)_{i=1,\ldots,d}$ d-dimensional Brownian motion
- pricing options, hedging,... $\longrightarrow$ calculation of $\mathrm{P}_t f = \mathrm{E}[f(X_t)]$ for some payoff f

Problems

- exotic option,
- infinite dimensional state space, nontrivial geometric structure,
- arbitrarily large d,
- unbounded operator in the drift term,
- payoff and vector fields V_i not necessarily bounded C_b^∞.

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- no closed form solution/formula for $\mathrm{P}_t f$,
- standard methods either do not work or exhibit serious problems.

Example: Heath-Jarrow-Morton equation

- state space: forward yield curves $r \colon [0,\infty) \to \mathbb{R}$, e.g. $H = \{r \in L^1_{\mathrm{loc}}(\mathbb{R}_+) |\, r' \in H^0_\alpha(\mathbb{R}_+)\}$
- HJM equation:

$$dr(t,r_0) = (Ar(t,r_0) + \alpha_{HJM}(r(t,r_0)))\,dt + \sum_{j=1}^{d} \sigma_j(r(t,r_0))dB_t^j$$

  - $A = \frac{d}{dx}$,
  - $r(0,r_0) = r_0$,
  - $\alpha_{HJM}(r)(x) = \sum_{j=1}^{d} \sigma_j(r)(x) \int_0^x \sigma_j(r)(\tau)\,d\tau$.
- Problems:
  - calibration: calibrate to the caplet prices observed on the market
  - pricing of swaptions

KLV approach

Kusuoka-Lyons-Victoir approach:

- abstract approach, free Lie algebra technique
  - all boils down to approximating $\exp(t(v_0 + \frac{1}{2}\sum_{i=1}^{d} v_i^2))$ in $\mathbb{R}\langle\langle v_0, \ldots, v_d \rangle\rangle$
- various types:
  - cubature schemes
  - Gaussian K-schemes:
    - Ninomiya-Victoir
    - Ninomiya-Ninomiya
- all KLV implementations respect the geometric structure of the state space
- under suitable assumption moving frame technique deals with unbounded operator A
- Gaussian K-schemes admit extrapolation of the Richardson type

Splitting-Semigroup approach

- operator semigroups: $\mathrm{P}_t = \exp(t(\tilde{V}_0 + \frac{1}{2}\sum_{i=1}^{d} V_i^2))$
- splitting up: $\tilde{V}_0 + \frac{1}{2}\sum_{i=1}^{d} V_i^2 = \sum_{i=0}^{k} A_i$
- splitting of classical order s:

$$Q_t f = \prod_{j=1}^{p} \prod_{m=0}^{k} \exp(\gamma_{j,m} t A_m) f$$

  - formal expansion in t of $P_t - Q_t$: no terms up to t^{s+1}
- Hansen, Osterman (2009): under some conditions: $\|\mathrm{P}_T f - Q_{T/n}^n f\| = O(n^{-s})$
- for $\gamma_{j,m} > 0$ numerical order s max. 2
- respects geometrical structure
- under suitable assumptions moving frame technique deals with unbounded operator A

Why splitting?

- simple to implement as a Monte Carlo algorithm
- two general ways how to do it:
  1. each $\exp(t\gamma_{j,m}A_m)f$ has an explicit solution
  2. $P_t^{(0)}f = \exp(t\tilde{V}_0)f$ and $P_t^{(i)}f = \exp(\frac{t}{2}V_i^2)f$, $i = 1, \ldots, d$
     - implementation:

$$\exp\left(\frac{t}{2}V_i^2\right)f(x) = E[\exp(\sqrt{t}ZV_i)f(x)], \quad Z \sim \mathcal{N}(0,1)$$

- usually combine the two ways $\rightarrow$ optimum performance

Need for acceleration

- Monte Carlo order of convergence $O(1/\sqrt{N}) \rightarrow$ want faster
- faster: quasi Monte Carlo, latices,... BUT need low integration dim.
- dim. of integration proportional to No. of timesteps

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- need high order approx. schemes:
  - high order $\Rightarrow$ less timesteps for same discretization err.

Extrapolation

Assumptions

- approximations $y_h^{(i)}$, $i \in I$ of y_0 amenable for all $h > 0$,
- known error expansion: $y_h^{(i)} = y_0 + \sum_{k=\alpha}^{\beta} c_k^{(i)} h^k + o(h^\gamma)$, $0 < \alpha \leq \beta \leq \gamma$ as $h \to 0$

$\downarrow$

Extrapolation

- killing error terms using lin. combination of:
  1. various approximations
  2. single approximation, varying h (Richardson extrapolation)

Extrapolation of splitting

- splitting for S(P)DE $\rightarrow$ num. order at most 2

- idea:
  1. calculate err. expansion in n: $P_t f - (Q^{(i)}_{t/n})^n f$ for various splittings
  2. use a combination of:
     - lin. combination of various splittings,
     - Richardson extrapolation

  to kill error terms/accelerate the numerical method

ODE solving

- generic case problems:
  - no explicit solution for $P_t^{(i)} f(x) \rightsquigarrow$ numerical ODE solver
  - compatibility: ODE solver $\leftrightarrow$ extrapolated splitting ???
  - num. order of ODE solver

- SDE case: Ninomiya, Ninomiya (2009)
- SPDE case: Dörsek, Veluščček (2012)
  - (extrapolated splitting, num order s) + (integration scheme ODE solver, num order $2s$) = OK

$\Downarrow$

- Runge-Kutta OK

Definiton of B^ψ spaces

- transplant of the idea to S(P)DE case: Röckner, Sobol (2006), Dörsek, Teichmann (2011)

Definition

- let X completely regular Hausdorff space
- $\psi\colon X \to \mathbb{R}$ admissible weight function if $\{x \in X | \, \psi(x) \leq R\}$ compact for all $R > 0$
- (X, ψ) weighted space

Definition

- let (X, ψ) weighted space
- $B^\psi(X) := \{f\colon X \to \mathbb{R} | \, \sup_{x \in X} \psi(x)^{-1} |f(x)| < \infty\}$
- $\|f\|_\psi := \sup_{x \in X} \psi(x)^{-1} |f(x)|$

B^{ψ} spaces II

- $(B^{\psi}(X), \|.\|_{\psi})$ is a Banach space
- instead of $\mathbb{R}$ one can use a Banach space
- $\mathcal{B}^{\psi}(X) :=$ closure of $C_b(X)$ in $B^{\psi}(X)$
- taking X dual of a separable Banach space, ψ a D–admisible weight function:
    - define $B_k^{\psi}(X)$ analogously, taking derivatives D^1 up to D^k into consideration
    - $\mathcal{B}_k^{\psi}(X_{w*}) :=$ closure of bounded smooth cylindrical functions in $B_k^{\psi}(X)$

Lie-Trotter splitting

- split SPDE into

$$\frac{d}{dt}z^0(t,x_0) = Az^0(t,x_0) + V_0(z^0(t,x_0)),$$
$$dz^j(t,x_0) = V_j(z^j(t,x_0)) \circ dB^j_t, \quad j = 1,\dots,d$$

- generated semigroups: P^i_t, $i = 0,\dots,d$
- their infinitesimal generators: $\mathcal{G}_i$, $i = 0,\dots,d$

- Lie-Trotter: $\overrightarrow{Q}^{LT}_t = P^0_t P^1_t \dots P^d_t$
- its adjoint: $\overleftarrow{Q}^{LT}_t = P^d_t P^{d-1}_t \dots P^0_t$

Numerical order of Lie-Trotter

Theorem (Dörsek, Teichmann (2011))

Given $f \in \mathcal{B}^{\psi^n_{\ell_0}}$:

- $P_t f \in \operatorname{dom} \mathcal{G}^2 \cap \bigcap_{j_1,j_2=1}^{d} \operatorname{dom} \mathcal{G}_{j_1} \mathcal{G}_{j_2}$
- $\sup_t \|\mathcal{G}_{j_1} \mathcal{G}_{j_2} P_t f\|_{\psi^n_{\ell_0}} < \infty$
- $\mathcal{G}^i P_t f = (\sum_{j=0}^{d} \mathcal{G}_j)^i P_t f, i = 1, 2$

$$\Downarrow$$

exists $C_f > 0$ *such that* $\forall t \in [0, T]$, $m \in \mathbb{N}$ *we have*
$\|P_t f - (\overrightarrow{Q}^{LT}_{t/m})^m f\|_{\psi^n_{\ell_0}} \leq C_f m^{-1}$

- the same holds for $\overleftarrow{Q}^{LT}_t$

Error expansion of Lie-Trotter splitting

Gyöngy, Krylov (2006): Assuming

- nested Banach spaces W_i, $i \geq 0$, W_1 dense in W_0, W_{i+1} continuously embeded in W_i, $i > 0$
- P_t and P_t^j bounded on W_i
- W_i invariant for P_t and P_t^j
- $\mathcal{G}_j(W_{i+1}) \subseteq W_i$, $\mathcal{G}_j|_{W_i}$ bounded
- ...

then $\forall f \in W_{2(m+1)}$ we have

$$(\overrightarrow{Q}^{LT}_{(T/n)})^n f - P_T f = \sum_{k=1}^{m} \overrightarrow{f}_k n^{-k} + \overrightarrow{r}_{m,n} n^{-m-1}$$

$$(\overleftarrow{Q}^{LT}_{(T/n)})^n f - P_T f = \sum_{k=1}^{m} \overleftarrow{f}_k n^{-k} + \overleftarrow{r}_{m,n} n^{-m-1}$$

where $\overrightarrow{f}_k, \overleftarrow{f}_k \in W_0$, $\|\overrightarrow{r}_{m,n}\|, \|\overleftarrow{r}_{m,n}\| \leq C_m$

Symmetrically weighted sequential splitting (SWSS)

- suitable choice of $\mathcal{B}_k^{\psi_\ell}(H_\ell)$ spaces satisfy the Gyöngy, Krylov (2006) condition
- Oshima, Teichmann, Velušček (2011) type of argument $\rightsquigarrow$ for every odd k: $\overrightarrow{f}_k = -\overleftarrow{f}_k$

$$\Downarrow$$

- symmetrically weighted seqential splitting:

$$Q_{T,n}^{SWSS} = \frac{1}{2}((\overrightarrow{Q}_{T/n}^{LT})^n + (\overleftarrow{Q}_{T/n}^{LT})^n)$$

- numerical order of $Q_{T,n}^{SWSS}$ is 2
- error expansion of $Q_{T,n}^{SWSS}$ contains only even terms

Extrapolation of SWSS

Theorem (Dörsek, Velušček 2011)

- $K_j \in \mathbb{N}$, $j = 1, \dots, m$ *pairwise distinct,*
- $\lambda_j \in \mathbb{R}$, $j = 1, \dots, m$ *such that* $\sum_{j=1}^m \lambda_j = 1$ *and* $\sum_{j=1}^m \lambda_j K_j^{-2i} = 0$, $i = 1, \dots, m$
- $f \in \mathcal{B}_{8m}^{\psi_\ell^{(n)}}((H_\ell)_w)$ *with* $0 \le \ell \le \ell_0 - 4m - 1$

then

$$\|P_T f - \sum_{j=1}^m \lambda_j Q_{T,nK_j}^{SWSS} f\|_{\mathcal{B}^{\psi_\ell^{(n)}}((H_\ell)_w)} \le C_f n^{-2m}$$

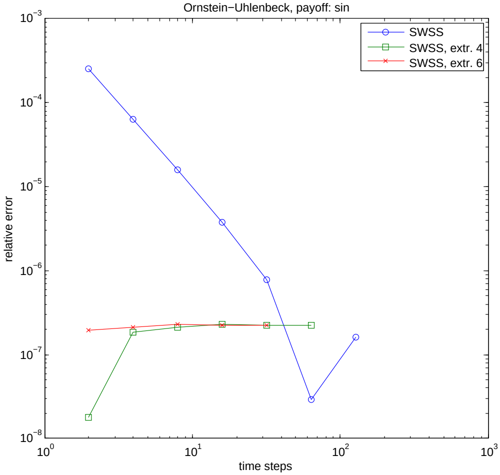
KLV perspective

- Lie-Trotter splittings $\overrightarrow{Q}_t^{LT}$ and $\overleftarrow{Q}_t^{LT}$ Gaussian K-schemes
- algebraic properties, (Kusuoka, 2009) $\rightsquigarrow$ first term of error expansion of $(\overrightarrow{Q}_{T/n}^{LT})^n$ and $(\overleftarrow{Q}_{T/n}^{LT})^n$ differs just by a sign

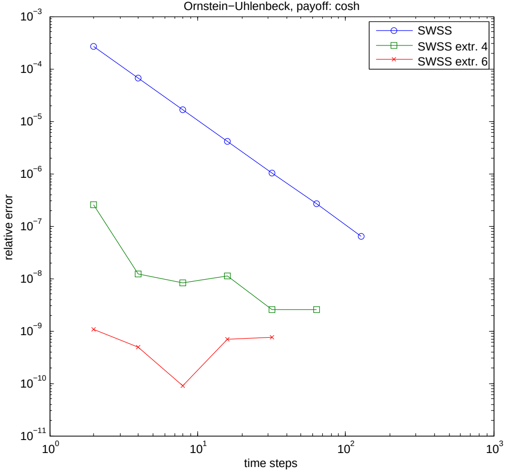
$$\Downarrow$$

- $Q_T^{SWSS} = \frac{1}{2}((\overrightarrow{Q}_{T/n}^{LT})^n + (\overleftarrow{Q}_{T/n}^{LT})^n)$ numerical order 2
- convergence in L^∞ norm $\rightsquigarrow$ SWSS also for BUC functions

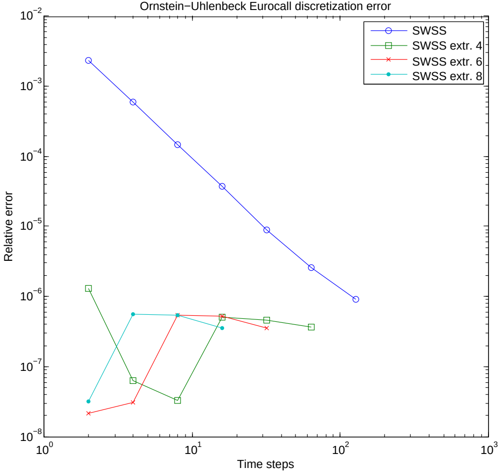
Ornstein-Uhlenbeck, payoff: sin



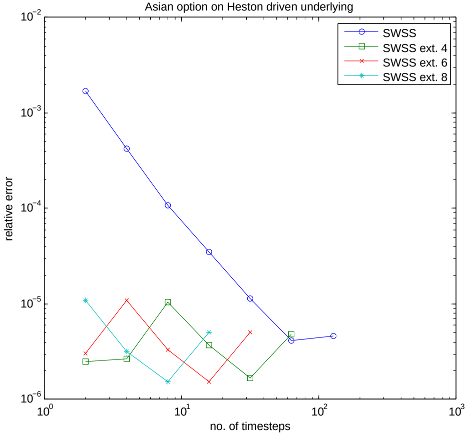
Ornstein-Uhlenbeck, payoff: cosh



Ornstein-Uhlenbeck, Eurocall



Heston model



- $T = 1$, $K = 1.05$, $\mu = 0.05$, $\alpha = 2$, $\beta = 0.1$, $\theta = 0.09$, $\rho = 0$, $X_0 = (1.0, 0.09)$
- $2\alpha\theta - \beta^2 > 0 \Longrightarrow$ volatility process=perturbation of a square of a Brownian motion

Open problems

- How to introduce ψ-controlled growth in Gaussian K-scheme setting without involving derivatives?
- further extrapolation of SWSS without the extra conditions on derivatives even in non-weighted case

Webpage

The C++ source code of the SWSS with examples and documentation available online:

www.math.ethz.ch/˜ doersekp/was

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