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Stochastic volatility for utility maximizers — A martingale approach

Simon Ellersgaard*

Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 Copenhagen, Denmark

Martin Tegnér†

Department of Engineering Science
Oxford-Man Institute of Quantitative Finance
University of Oxford
Parks Road, OX1 3PJ Oxford, England

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Abstract

Using Martingale methods, we study the problem of optimal consumption-investment strategies in a complete financial market characterized by stochastic volatility. With Heston's model as the working example, we derive optimal strategies for a constant relative risk aversion (CRRA) investor with particular attention to the cases where (i) she solely seeks to optimize her utility for consumption, and (ii) she solely seeks to optimize her bequest from investing in the market. Furthermore, we test the practical utility of our work by conducting an empirical study based on real market-data from the S&P500 index. Here, we concentrate on wealth maximization and investigate the degree to which the inclusion of derivatives facilitates higher welfare gains. Our experiments show that this is indeed the case, although we do not observe realized wealth-equivalents as high as expected. Indeed, if we factor in the increased transaction costs associated with including options, the results are somewhat less convincing.

Keywords: Merton's portfolio problem; stochastic volatility: Heston model; Martingale approach.

†Corresponding author.
Email addresses: *ellersgaard@math.ku.dk, †martin.tegner@eng.ox.ac.uk

1. Utility Maximization and Stochastic Volatility

1.1. Introduction

Since at least the early 1960s a substantial body of empirical finance literature has systematically documented the stochasticity of asset price volatility (see Chewlow and Xu, 1993). Indeed, from fat-tailed returns with time-varying fluctuations (see Poon (2005)) to volatility smiles in the options landscape (see Cont and Tankov (2004)), the constancy of volatility employed by Black and Scholes (1973) has without doubt been established as a simplifying assumption. On the modeling front, a plethora of possible resolutions have been proposed to remedy this uncomfortable fact: most prominently *local volatility models* in which σ is a deterministic function of the random stock price (see e.g., Dupire (1994) and Derman and Kani (1994)), and *diffusion-based stochastic volatility models* in which σ is modeled directly as a stochastic differential equation (see e.g., Heston (1993), Carr and Sun (2007), and Hagan *et al.* (2002)). Both approaches must be considered as significant steps towards designing calibratable models to observed market phenomena, although neither can be said to be void of imperfections. However, the latter is arguably the more sophisticated of the two, being as it were more readily susceptible to theoretical augmentation. Derivatives pricing likewise becomes a matter of some interest: whilst local volatility models will have us believe that options are perfectly hedgeable using bonds and the underlying stocks, thereby making them formally redundant, this is not so for valuation under stochastic volatility models. Here, incompleteness forces us to make further exogenous assumptions about the behavior of the market in order to pin down our risk neutral measure, $\mathbb{Q}$. Specifically, to value one option, enough similar traded options must already exist in the market, in order for us to say anything concrete (in somewhat more abstract terms: a supply-and-demand induced market price of risk must prevail — see e.g., Björk, 2009, Chap. 15).

Surprisingly, while stochastic volatility constitutes a major research area in the pricing realm of the quant-finance community, relatively few papers deal with its impact on continuous time portfolio optimization as championed by Merton (1969). In fact, to the best of our knowledge, the first authors to deal explicitly with the issue are Liu and Pan, a little more than a decade ago. The more cursory of their analyses is found in Liu (2007), in which bequest optimization in a Heston-driven[1] bond-stock economy is used to illustrate a grander theoretical point about solutions to HJB equations. Briefly, under the assumption that the market price of risk is proportional to volatility, $\lambda_1 = \bar{\lambda}_1\sigma$, Liu shows that the optimal portfolio weight to be placed on the stock by a rational constant relative risk

[1]See Heston (1993) and Sec. 4 for an exposition of this model.

aversion (CRRA) investor is of the form

$$\pi_{S,t}^{\text{Liu}} = \frac{\bar{\lambda}_1}{\gamma} - \rho\sigma_v \frac{\gamma - 1}{\gamma} L(T - t), \tag{1}$$

where $\bar{\lambda}_1/\gamma$ is the Merton ratio, and the second term is a stochastic volatility *hedge correction* in which L is the deterministic function $L(\tau) = \frac{\bar{\lambda}_1^2}{\gamma} \frac{(e^{\eta\tau}-1)}{(\epsilon+\eta)(e^{\eta\tau}-1)+2\eta}$, where $\epsilon \equiv \kappa + \frac{\gamma-1}{\gamma}\rho\sigma_v\bar{\lambda}_1$, $\eta \equiv (\epsilon^2 + \frac{\gamma-1}{\gamma^2}\sigma_v^2(\rho^2 + \gamma[1-\rho^2])\bar{\lambda}_1^2)^{1/2}$, and κ, σ_v, ρ are Heston parameters (see Eq. (38)). Throughout this paper, we refer to this result as Liu's strategy. Little is said by Liu on the empirical implications, yet it is well-known that the correction hedge is negligible. For example, our own investigation (Ellersgaard and Tegnér, 2015) reveals that the hedge correction is multiple orders of magnitude smaller than the Merton weight for realistic parameter specifications, thus leading to non-measurable improvements in the investor's welfare.

A far richer theoretical account of the role of stochastic volatility in portfolio maximization is provided in Liu and Pan (2003), which extends the above framework to include jumps in the underlying price process, and which completes the market by including tradeable derivative securities (specifically, a straddle, chosen for its sensitivity to volatility risk). By solving the relevant HJB equation, optimal portfolio weights are provided (in terms of certain partial derivatives); moreover, through the employment of market calibrated parameters, Liu and Pan estimate that the primary demand for derivatives is nested in the myopic component of the portfolio weight (rather than the volatility hedge correction). Based on the same parameters, they also establish significant improvements in *certainty equivalent wealth* through the act of including derivatives in a utility maximized portfolio.

Other noteworthy contributions to the literature include Kraft (2005), Branger and Hansis (2012), and Chacko and Viceira (2005). Kraft notably establishes a formal verification argument for Liu's formula (qua the CIR nature of the Heston dynamics this is non-trivial). Branger and Hansis consider a variation of Liu and Pan in which optimal "buy-and-hold" strategies are studied for an isoelastic (CRRA) investor who can also trade in a stock option. Here, the market is again Hestonian, albeit with the interesting caveat that they allow for correlated jumps in the driving processes of the stock price and the variance. Fundamentally, this paper seeks to uncover the utility gains obtained through the incorporation of derivatives and the losses incurred due to the omission of risk factors and erroneous estimates of risk premia. Finally, Chacko and Viceira scrutinize the problem of establishing an optimal consumption process for an investor with Epstein–Zin utility under stochastic volatility. Working with what effectively translates to a 3/2-model in a stock and money account economy, Chacko and Viceira manage to derive closed-form approximations to the optimal consumption level, followed by a perspective from empirics.

1.2. Overview

Our intention with this paper is two-fold. First, we seek to establish a unified framework for stochastic volatility utility maximizers who care about their continuous rate of consumption *as well as* their terminal wealth (Sec. 2). We do so from the perspective of complete-market Martingale theory[2] under a fairly generic set-up (Sec. 3), thus distinguishing ourselves from the prevailing literature in our area in which the Bellman approach is otherwise ubiquitous. Specifically, we solve the optimization problem through Lagrangian optimization principles, and show that the optimal consumption-investment policies a rational agent should pursue can be determined by solving a linear partial differential equation (PDE). We instantiate these findings with an old workhorse in stochastic-volatility pedagogy, the Heston model, which under suitable assumptions regarding the market price of risk is found to yield an exponential affine solution to the aforementioned PDE. This in turn facilitates a comparatively straightforward discussion of the conceptual implications of incorporating a random variance component in utility optimization (Sec. 4). Nevertheless, we stress that our framework is sufficiently rich to accommodate "less trivial" stochastic volatility models. In particular, the establishment of closed form expressions for the 3/2 model is quite possible, based on manipulations not too different from those highlighted in Ellersgaard and Tegnér (2015).

Second, we endeavor to provide a novel empirical study based on the body of theory highlighted above (Sec. 5). We do so by tracking the performance of a wealth-maximizing investor who trades in a Hestonian market comprised of a bond, a stock and a European option. Specifically, using the optimal Heston controls, we scrutinize an automated trading experiment where 15 years' of real market data from the LIBOR rate, S&P500 options, and the underlying index constitute the investment space. Fundamentally, we here seek to verify the oft reported result that the inclusion of options into the investment portfolio has the potential of increasing the wealth of the investor — at least when looking at *expected* levels of certainty equivalents in wealth (see e.g., Larsen and Munk (2012), and Liu and Pan (2003)). For market-calibrated parameters we are indeed able to corroborate such a value gain: with daily trading in the bond and S&P500 index, we see an increase in *realized* certainty equivalents from including either a call or a straddle. This means that the bond-stock-derivative investor can initiate

[2]For a survey see e.g., Björk (2009, Chap. 20).

her portfolio with a few percentage points lesser capital and still obtain a realized utility equivalent to that of the optimal portfolio without options. As expected, the realized certainty equivalent increases with a user-specified (negative) market price of volatility risk. However, we fall considerably short of reaching the promising predicted gains implied from the highly negative levels of the market price. Yet, for the most profitable specification, the investor is better off from trading options — almost regardless of when the investor decides to terminate the portfolio during the 15 years' period.

2. Problem Set-up

2.1. Market assumptions

Following the path betrodden by Black and Scholes (1973), and Merton (1973), we start out by considering a financial market which is frictionless, arbitrage free, and allows for continuous trading. Three assets which jointly complete the market are assumed to prevail, viz. a risk-free money account (a *bond*), one fundamental risky security (a *stock*), and one derivative security with a European exercise feature at time $t = T'$. We define the dynamical equations of these securities by first introducing the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T']}, \mathbb{P})$, where Ω represents all possible states of the economy, $\mathbb{P}$ is the real-world probability measure, and $\mathscr{F}_t$ is the augmented natural filtration of the bivariate standard Brownian motion $\boldsymbol{W} = (W_1, W_2)^\intercal$. With this in mind, we specify the price process dynamics of the money account $\{B_t\}_{t\in[0,T']}$ as the deterministic equation, $dB_t = rB_t dt$, where $B_0 = b_0 \in \mathbb{R}_+$ and r is the constant rate of interest. As for the risky security with price process $\{S_t\}_{t\in[0,T']}$, we posit a generic stochastic volatility model

$$\begin{aligned} dS_t &= \mu_S(t, V_t)S_t dt + \sqrt{V_t} S_t dW_{1t}, \\ dV_t &= \alpha(t, V_t)dt + \beta(t, V_t)(\rho dW_{1t} + \sqrt{1-\rho^2} dW_{2t}), \end{aligned} \tag{2}$$

where $(S_0, V_0) = (s_0, v_0) \in \mathbb{R}^2_+$, and $\{V_t\}_{t\in[0,T']}$ is the variance process which we assume strictly positive. As for the dynamical constituents: μ_S, α, and β are taken to be real-valued deterministic functions $[0, T'] \times \mathbb{R}_+ \mapsto \mathbb{R}$, whilst $\rho = \mathbb{C}\mathrm{orr}[dS_t, dV_t] \in (-1, 1)$, is a Pearson correlation coefficient between the stock and variance processes. Finally, as for the European derivative, we envision a one-time pay-off $\Phi(S_{T'})$ based on the magnitude of the stock value at expiry time T'. Letting $\{D_t\}_{t\in[0,T']} = \{D(t, S_t, V_t)\}_{t\in[0,T']}$ represent the price process of the derivative, it follows from Itô's lemma that:

$$dD_t = \mu_D(t, S_t, V_t)D_t dt + \sigma_{1D}(t, S_t, V_t)D_t dW_{1t} + \sigma_{2D}(t, S_t, V_t)D_t dW_{2t}, \tag{3}$$

where $D_{T'} = \Phi(S_{T'})$, and $\mu_D, \sigma_{1D}, \sigma_{2D} : [0, T'] \times \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}$ are the functions

$$\mu_D(t,s,v) \equiv D^{-1}\left[\partial_t D + \mu_S(t,v)s\partial_s D + \alpha(t,v)\partial_v D + \frac{1}{2}vs^2\partial^2_{ss}D \right.$$
$$\left. + \frac{1}{2}\beta^2(t,v)\partial^2_{vv}D + \rho\beta(t,v)\sqrt{v}s\partial^2_{sv}D\right], \tag{4a}$$

$$\sigma_{1D}(t,s,v) \equiv D^{-1}[\rho\beta(t,v)\partial_v D + \sqrt{v}s\partial_s D], \tag{4b}$$

$$\sigma_{2D}(t,s,v) \equiv D^{-1}[\sqrt{1-\rho^2}\beta(t,v)\partial_v D], \tag{4c}$$

assuming that $D \in \mathcal{C}^{1,2,2}([0,T'], \mathbb{R}^2_+)$.

Crucial to our derivations in the subsequent sections, we now enforce the following minimal structure upon the *aggregate* risk preferences of agents trading in our tripartite economy.

Assumption 1. The market prices of risk $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)^\intercal$ associated with $\boldsymbol{W}$ is a deterministic function of v only. In concrete terms this means that

$$\lambda_1(v) = \frac{\mu_S(t,v) - r}{\sqrt{v}}, \tag{5}$$

and

$$\lambda_2(v) = \frac{\mu_D(t,s,v) - r}{\sigma_{2D}(t,s,v)} - \frac{\sigma_{1D}(t,s,v)}{\sigma_{2D}(t,s,v)}\lambda_1(v). \tag{6}$$

We call this the weak Heston assumption for reasons which will become clearer below.[3]

Now, from (5) we may define the risk-neutral measure $\mathbb{Q}$ on $\mathcal{F}_{T'}$ through the Radon–Nikodym derivative

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{T'} \equiv \exp\left\{-\frac{1}{2}\int_0^{T'} ||\boldsymbol{\lambda}(V_t)||^2 dt - \int_0^{T'} \boldsymbol{\lambda}^\intercal(V_t)d\boldsymbol{W}_t\right\}, \tag{7}$$

where $||\cdot||$ is the Euclidian norm. For reasons of brevity, we shall henceforth refer to (7) simply as $\xi_{T'}$; indeed, we may consider defining the general process ξ_t for $t \in [0, T']$. Assuming the validity of Novikov's condition, it follows that

[3]We emphasize that this is not a vacuous statement: specifically, the weak Heston assumption is not a *gauge freedom*, as it invariably does say *something* about supply and demand in the market. On the other hand, it is not an approximation either: clearly, we have the mathematical freedom to suppose whatever we want here.

$\mathbb{E}[\xi_t] = 1$, whence $\mathbb{Q}$ is an equivalent local Martingale measure (ELMM).[4] All discounted asset prices under $\mathbb{Q}$ are therefore local Martingales, which can be verified by combining Girsanov's transformation

$$dW_{it} = -\lambda_i(V_t)dt + dW_{it}^{\mathbb{Q}},$$

for $i = 1, 2$ with the price dynamics (2) and (3). Finally, upon combining the market price of risk (6) with the Itô expressions (4) we readily find that the PDE governing the price of the derivative is of the form

$$0 = \partial_t D + rs\partial_s D + \{\alpha(t,v) - \beta(t,v)[\rho\lambda_1(v) + \sqrt{1-\rho^2}\lambda_2(v)]\}\partial_v D + \frac{1}{2}vs^2\partial_{ss}^2 D + \frac{1}{2}\beta^2(t,v)\partial_{vv}^2 D + \rho\beta(t,v)\sqrt{v}s\partial_{sv}^2 D - rD, \tag{8}$$

subject to the terminal condition $D(T', s, v) = \Phi(s)$.

2.2. Investor assumptions

We consider the case of an investor who trades in the three assets in a self-financing manner over the temporal horizon $[0, T] \subseteq [0, T']$, with the intention of maximizing the expected discounted utility of her continuous rate of consumption, c, as well as her terminal wealth, $\mathscr{W}_T$. Specifically, we are interested in determining the optimal consumption process $\{c_s^*\}_{s\in[0,T]}$, alongside the optimal portfolio weights $\{\boldsymbol{\pi}_t^*\}_{t\in[0,T]} \equiv \{\pi_{S,t}^*, \pi_{D,t}^*\}_{s\in[0,T]}$ which the investor should place on the stocks and the derivative[5] that solve the optimization problem

$$\sup_{\{c_t, \boldsymbol{\pi}_t\}_{t\in[0,T]}\in\mathscr{A}} \mathbb{E}\left[\alpha\int_0^T e^{-\delta s}u(c_s)ds + (1-\alpha)e^{-\delta T}u(\mathscr{W}_T)\right], \tag{9}$$

where the wealth process $\{\mathscr{W}_t\}_{t\in[0,T]}$ is described by the *self-financing condition*

$$d\mathscr{W}_t = \mathscr{W}_t[r + \boldsymbol{\pi}_t^\top \boldsymbol{\Sigma}_t \boldsymbol{\lambda}(V_t)]dt - c_t dt + \mathscr{W}_t \boldsymbol{\pi}_t^\top \boldsymbol{\Sigma}_t d\mathbf{W}_t, \tag{10}$$

with initial condition $\mathscr{W}_0 = w_0 \in \mathbb{R}^+$.

[4]We assume the Novikov condition in order to establish the existence of the ELMM. Rather remarkably, existence is something all too often glossed over in the stochastic volatility literature. Indeed, there are somewhat spectacular examples of stochastic volatility models where the no-arbitrage condition generally breaks down cf. e.g., the Stein & Stein model. Furthermore, it is reasonable to show formally that discounted asset prices are *true Martingales* as opposed to strictly local ones. Although failure of the true Martingale property does not entail arbitrage, it can lead to peculiarities such as the breakdown of put-call parity. For examples and general theory pertaining to these fascinating issues we refer the reader to Wong and Hyde (2006).

[5]We assume that the remaining fraction of the wealth $\pi_{B,t} = 1 - \pi_{S,t} - \pi_{D,t}$ is allocated to the risk-free money account.

Here, $\alpha \in [0,1]$ is a relative importance weight between continuous consumption and bequest, $\delta \in \mathbb{R}_+$ is a subjective discounting factor, and $u : \mathbb{R}^+ \mapsto \mathbb{R}$ is a utility function, which we take to be of the CRRA variety: $u(x) = x^{1-\gamma}/(1-\gamma)$, where $\gamma \in \mathbb{R}_+\backslash\{1\}$ is the investor's level of risk aversion. Formally, we require that the consumption process $c : [0,T] \times \Omega \mapsto \mathbb{R}_+$ and the portfolio process $\boldsymbol{\pi} = (\pi_1, \pi_2)^\intercal : [0,T] \times \Omega \mapsto \mathbb{R}^2$ are $\mathscr{F}_t$-progressively measurable processes which lie in the admissibility set $\mathscr{A} \equiv \{(c, \boldsymbol{\pi}) : \int_{[t,T]} (c_s + \boldsymbol{\pi}_s^\intercal \boldsymbol{\pi}_s) ds < \infty \,\&\, \mathscr{W}_t \geq 0 \text{ a.s. } \forall t \in [0,T]\}$. In particular, no restrictions on short-selling and leveraging are enforced, whilst doubling strategies are ruled out. Finally, $\boldsymbol{\Sigma}$ is the matrix-valued volatility process associated with the two risky assets, S and D, viz.

$$\boldsymbol{\Sigma}_t \equiv \begin{pmatrix} \sqrt{V_t} & 0 \\ \sigma_{1D}(t, S_t, V_t)\rho & \sigma_{2D}(t, S_t, V_t)\sqrt{1-\rho^2} \end{pmatrix},$$

where σ_{1D} and σ_{2D} are given in (4).

3. The Martingale Solution

3.1. The optimal wealth process

Whilst (9), in principle, can be solved dynamically by translating the problem into a Hamilton–Jacobi–Bellman equation, there is an added level of complexity introduced by the possibly non-Lipschitzian nature of the driving stochastic differential equations. Specifically, we can no longer rely on standard verification arguments as found in references such as Pham, and Ross. This is clearly problematic to the extent that classical volatility models such as Heston's CIR dynamics for the variance manifestly do not satisfy Lipschitz' conditions. While the added complications here may not be insurmountable, it does introduce a level of clumsiness into what we desire to be a generic framework for utility maximizing agents. (For a rigorous verification argument pertaining to utility optimization in a Hestonian economy, the reader is referred to Kraft (2005).)

Thus, to avoid these technicalities, we thus opt for a Martingale procedure instead. Specifically, the observation is here that Eqs. (9) and (10) may be recast as the static optimisation problem

$$\sup_{\{\{c_t\}_{t\in[0,T]}, \mathscr{W}_T\}} \mathbb{E}\left[\alpha \int_0^T e^{-\delta s} u(c_s) ds + (1-\alpha) e^{-\delta T} u(\mathscr{W}_T)\right], \tag{11}$$

in which we optimize over consumption and the terminal wealth directly subject to the natural Martingale constraint that $\mathbb{Q}$ discounted future cashflows equate the present wealth level: $w_0 = \mathbb{E}^{\mathbb{Q}}[\int_0^T e^{-rs} c_s ds + e^{-rT}\mathscr{W}_T]$. The latter is commonly referred to as the *budget equation*, and is found rigorously demonstrated in Munk (2013). Jointly, this can of course be written on Lagrangian form as

$$\mathscr{L} = \mathbb{E}\bigg[\alpha \int_0^T e^{-\delta s} u(c_s) ds + (1-\alpha) e^{-\delta T} u(\mathscr{W}_T) \\ - \eta \bigg\{ \int_0^T e^{-rs} \xi_s c_s ds + e^{-rT} \xi_T \mathscr{W}_T \bigg\}\bigg],$$

where η is a Lagrange multiplier and ξ is the Radon–Nikodym derivative we introduced (7), thus securing that the expectation is taken under the common measure, $\mathbb{P}$. Differentiating $\mathscr{L}$ partially with respect to c_t and $\mathscr{W}_T$ and equating to zero, we find the following expressions for the optimal rate of consumption and the optimal terminal wealth:

$$c_t^* = I\left(\frac{\eta e^{(\delta-r)t} \xi_t}{\alpha}\right), \tag{12a}$$

$$\mathscr{W}_T^* = I\left(\frac{\eta e^{(\delta-r)T} \xi_T}{1-\alpha}\right), \tag{12b}$$

where we have defined the *inverse marginal utility function* $I(x) = x^{-1/\gamma}$. To rid our expressions of the multiplier η, we substitute (12) into the $\mathbb{P}$-form of the budget constraint. After a few manipulations we find

$$\eta^{-1/\gamma} = \frac{w_0}{\alpha^{1/\gamma} \int_0^T e^{-(r+q)s} \mathbb{E}[\xi_s^{1-1/\gamma}] ds + (1-\alpha)^{1/\gamma} e^{-(r+q)T} \mathbb{E}[\xi_T^{1-1/\gamma}]},$$

where we have used Fubini's theorem alongside the shorthand notation $q \equiv (\delta - r)/\gamma$. Substituting this expression back into (12) we thus have

$$c_t^* = \frac{\alpha^{1/\gamma} e^{-qt} \xi_t^{-1/\gamma} w_0}{G(0,v)}, \tag{13a}$$

$$\mathscr{W}_T^* = \frac{(1-\alpha)^{1/\gamma} e^{-qT} \xi_T^{-1/\gamma} w_0}{G(0,v)}, \tag{13b}$$

where we, for for reasons that will shortly become apparent, have introduced the function $G : [0,T] \times \mathbb{R}^+ \mapsto \mathbb{R}$

$$G(t,v) = \alpha^{1/\gamma} \int_t^T e^{-(r+q)(s-t)} H^s(t,v) ds + (1-\alpha)^{1/\gamma} e^{-(r+q)(T-t)} H^T(t,v), \tag{14}$$

where $H^s : [0,T] \times \mathbb{R}^+ \mapsto \mathbb{R}$ is a class of functions defined for each upper temporal value $s \in [t,T]$ as the $\mathbb{P}$-expectation of the ratio $(\xi_s/\xi_t)^{1-1/\gamma}$ conditional on the information $(t, V_t) = (t, v)$,

$$H^s(t,v) = \mathbb{E}_{t,v}\left[\left(\frac{\xi_s}{\xi_t}\right)^{1-1/\gamma}\right]. \tag{15}$$

From the definition of ξ_s, (7), it quickly becomes apparent that the quantity $\xi_s^{1-1/\gamma}$ in itself has the *resemblance* of a Radon–Nikodym derivative, ξ_s^0, viz. that of

$$\left.\frac{d\mathbb{Q}^0}{d\mathbb{P}}\right|_s \equiv \exp\left\{-\frac{1}{2}(1-1/\gamma)^2\int_0^s ||\boldsymbol{\lambda}(V_u)||^2 du - (1-1/\gamma)\int_0^s \boldsymbol{\lambda}^\intercal(V_u) d\boldsymbol{W}_u\right\}. \tag{16}$$

In fact, one may readily check that $\xi_s^{1-1/\gamma}$ and ξ_s^0 are related through $\xi_s^{1-1/\gamma} = \xi_s^0 \exp\{\frac{1-\gamma}{2\gamma^2}\int_0^s ||\boldsymbol{\lambda}(V_u)||^2 du\}$. Substituting this into (15) one may thence re-express the H^s-function under the alternative measure $\mathbb{Q}^0$

$$H^s(t,v) = \mathbb{E}_{t,v}^{\mathbb{Q}^0}\left[\exp\left\{\frac{1-\gamma}{2\gamma^2}\int_t^s ||\boldsymbol{\lambda}(V_u)||^2 du\right\}\right]. \tag{17}$$

Note that the explicit dependence on the Wiener increments effectively has been suppressed through this second change of measure.

To see how this comes in handy, let us determine the optimal wealth process $\mathscr{W}_t^*$ for all times $t \in [0, T]$. Starting from the time t budget equation,

$$\begin{aligned}
\mathscr{W}_t^* &= \mathbb{E}_{t,v}^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)} c_s^* ds + e^{-r(T-t)} \mathscr{W}_T^*\right] \\
&= \frac{1}{\xi_t}\mathbb{E}_{t,v}\left[\int_t^T e^{-r(s-t)} \xi_s c_s^* ds + e^{-r(T-t)} \xi_T \mathscr{W}_T^*\right] \\
&= \frac{1}{\xi_t}\mathbb{E}_{t,v}\left[\int_t^T e^{-r(s-t)} \xi_s \frac{\alpha^{1/\gamma} e^{-qs} \xi_s^{-1/\gamma} w_0}{G(0,v)} ds \right.\\
&\qquad \left. + e^{-r(T-t)} \xi_T \frac{(1-\alpha)^{1/\gamma} e^{-qT} \xi_T^{-1/\gamma} w_0}{G(0,v)}\right] \\
&= \frac{w_0 e^{-qt} \xi_t^{-1/\gamma}}{G(0,v)} \mathbb{E}_{t,v}\left[\alpha^{1/\gamma}\int_t^T e^{-(r+q)(s-t)} \left(\frac{\xi_s}{\xi_t}\right)^{1-1/\gamma} ds \right.\\
&\qquad \left. + (1-\alpha)^{1/\gamma} e^{-(r+q)(T-t)} \left(\frac{\xi_T}{\xi_t}\right)^{1-1/\gamma}\right] \\
&= \frac{w_0 e^{-qt} \xi_t^{-1/\gamma}}{G(0,v)} \alpha^{1/\gamma} \int_t^T e^{-(r+q)(s-t)} \mathbb{E}_{t,v}\left[\left(\frac{\xi_s}{\xi_t}\right)^{1-1/\gamma}\right] ds
\end{aligned}$$

$$+ (1-\alpha)^{1/\gamma} e^{-(r+q)(T-t)} \mathbb{E}_{t,v}\left[\left(\frac{\xi_T}{\xi_t}\right)^{1-1/\gamma}\right]$$

$$= \frac{w_0 e^{-qt}\xi_t^{-1/\gamma}}{G(0,v)} \left\{\alpha^{1/\gamma} \int_t^T e^{-(r+q)(s-t)} H^s(t,v) ds \right.$$

$$\left. + (1-\alpha)^{1/\gamma} e^{-(r+q)(T-t)} H^T(t,v)\right\},$$

where the second equality makes use of the Abstract Bayes' formula,[6] the third equality introduces the optimal consumption and wealth process expressions as determined in (13), the fourth equality uses the $\mathscr{F}_t$ measurability of $\xi_t^{-1/\gamma}$, the fifth equality uses Fubini's theorem, and the sixth line uses the definition of the H^s-function (15). Hence, using the definition of G, (14), it follows that the optimal wealth process is of the form

$$\mathscr{W}_t^* = \frac{w_0 e^{-qt}\xi_t^{-1/\gamma} G(t,v)}{G(0,v)}. \tag{18}$$

3.2. Notes on the functions H and G

Having derived an identity of the optimal wealth process, (18), one may reasonably ponder whether we have established anything of merit judged from a purely computational perspective. In this section, we will drive home the message that the answer is in the affirmative. Specifically, we establish the differential forms of H and G and show that they both satisfy partial linear differential equations, thus alleviating the potentially cumbersome search for analytical or numerical solutions. Indeed, upon comparing these steps with the non-linear Hamilton–Jacobi–Bellman equation, it is evident that we have made progress.

Starting with the first item, since the $H_t^s = H^s(t,v)$ is assumed to be a function of time and variance only, it follows from Itô's lemma and the dynamics (2) that:

$$dH_t^s = \mu_{H^s}(t,V_t)H_t^s dt + \sigma_{1H^s}(t,V_t)H_t^s dW_{1t} + \sigma_{2H^s}(t,V_t)H_t^s dW_{2t}, \tag{19}$$

where

$$\mu_{H^s}(t,v) \equiv (H^s)^{-1}\left[\partial_t H_t^s + \alpha(t,v)\partial_v H_t^s + \frac{1}{2}\beta^2(t,v)\partial_{vv}^2 H_t^s\right], \tag{20a}$$

$$\sigma_{1H^s}(t,v) \equiv (H^s)^{-1}\rho\beta(t,v)\partial_v H_t^s, \tag{20b}$$

$$\sigma_{2H^s}(t,v) \equiv (H^s)^{-1}\sqrt{1-\rho^2}\beta(t,v)\partial_v H_t^s. \tag{20c}$$

[6]See Björk (2009), Proposition B.41.

Now, from Girsanov's theorem, the $\mathbb{Q}^0$-Brownian increments are related to the $\mathbb{P}$-Brownian increments through

$$dW_{it} = -(1 - 1/\gamma)\lambda_i(V_t)dt + dW_{it}^{\mathbb{Q}^0},$$

for $i = 1, 2$. Substituting these into the dynamics (2) for the variance V_t, we find that its drift changes as $\alpha(t, v) \mapsto \alpha^0(t, v)$ where

$$\alpha^0(t, v) \equiv \alpha(t, v) - (1 - 1/\gamma)\beta(t, v)[\rho\lambda_1(v) + \sqrt{1 - \rho^2}\lambda_2(v)].$$

Thus, using Feynman–Kac's theorem we may deduce that H^s solves the linear PDE

$$0 = \partial_t H^s + \alpha^0(t, v)\partial_v H^s + \frac{1}{2}\beta^2(t, v)\partial_{vv}^2 H^s + \frac{1-\gamma}{2\gamma^2}||\boldsymbol{\lambda}(v)||^2 H^s, \tag{21}$$

subject to the terminal condition $H^s(s, v) = 1$.

A similar analysis can be applied to the G-function directly. Since this involves applying Itô's lemma to a stochastic function nested under a time integral, we invoke the *generalized Leibniz rule* which extends Leibniz' well-known integral rule to stochastic processes (see Heath *et al.* (1992) or Munk and Sørensen (2004)). Specifically, it can be shown that

$$dG_t = \mu_G(t, V_t)G_t dt + \sigma_{1G}(t, V_t)G_t dW_{1t} + \sigma_{2G}(t, V_t)G_t dW_{2t}, \tag{22}$$

where

$$\begin{aligned}\mu_G(t, v) \equiv (G)^{-1}\Bigg[&\alpha^{1/\gamma}\left(\int_t^T \{r + q + \mu_{H^s}\}e^{-(r+q)(s-t)}H^s ds - 1\right)\\ &+ (1-\alpha)^{1/\gamma}\{r + q + \mu_{H^T}\}e^{-(r+q)(T-t)}H^T\Bigg],\end{aligned} \tag{23a}$$

$$\begin{aligned}\sigma_{iG}(t, v) \equiv (G)^{-1}\Bigg[&\alpha^{1/\gamma}\int_t^T e^{-(r+q)(s-t)}\sigma_{iH^s}H^s ds\\ &+ (1-\alpha)^{1/\gamma}e^{-(r+q)(T-t)}\sigma_{iH^T}H^T\Bigg],\end{aligned} \tag{23b}$$

for $i = 1, 2$. Furthermore, upon pulling the expectation operator out from under the integral in (14) and rewriting the expression in terms of measure $\mathbb{Q}^0$ it again follows from Feynman–Kac that:

$$\begin{aligned}0 = &\partial_t G + \alpha^0(t, v)\partial_v G + \frac{1}{2}\beta^2(t, v)\partial_{vv}^2 G\\ &+ \left\{\frac{1-\gamma}{2\gamma^2}||\boldsymbol{\lambda}(v)||^2 - (r + q)\right\}G + \alpha^{1/\gamma},\end{aligned} \tag{24}$$

subject to the terminal condition $G(T, v) = (1-\alpha)^{1/\gamma}$. Nevertheless, this expression should be seen as an aside. For computational purposes, it is clearer to rewrite G in terms of H through (14), which in turn can be solved through (33).

3.3. The optimal consumption-investment policies

Finally, we are in a position to determine the optimal consumption-investment strategies a rational investor should pursue at the face of stochastic volatility. First, upon combining the optimal wealth process (18) with the expression for c_t^* in (13), we find that the optimal rate of consumption can be written on the form

$$c_t^* = \alpha^{1/\gamma} \frac{\mathscr{W}_t^*}{G(t, v)}. \tag{25}$$

As for optimal portfolio weights, π_{St}^* and π_{Dt}^*, an application of Itô's lemma to (18) reveals that

$$\begin{aligned} d\mathscr{W}_t^* &= -qe^{-qt}w_0 \frac{G_t}{G_0}\xi_t^{-1/\gamma}dt + e^{-qt}w_0\frac{1}{G_0}\xi_t^{-1/\gamma}dG_t + e^{-qt}w_0\frac{G_t}{G_0}d(\xi_t^{-1/\gamma}) \\ &\quad + e^{-qt}w_0\frac{1}{G_0}dG_t d(\xi_t^{-1/\gamma}) \\ &= \text{drift} + e^{-qt}w_0\frac{1}{G_0}\xi_t^{-1/\gamma}[\sigma_{1G}G_t dW_{1t} + \sigma_{2G}G_t dW_{2t}] \\ &\quad - e^{-qt}w_0\frac{G_t}{G_0}\frac{1}{\gamma}\xi_t^{-1/\gamma-1}d\xi_t \\ &= \text{drift} + \mathscr{W}_t^*[\sigma_{1G}dW_{1t} + \sigma_{2G}dW_{2t}] \\ &\quad + e^{-qt}w_0\frac{G_t}{G_0}\frac{1}{\gamma}\xi_t^{-1/\gamma-1}\xi_t[\lambda_1 dW_{1t} + \lambda_2 dW_{2t}] \\ &= \text{drift} + \mathscr{W}_t^*[\sigma_{1G}dW_{1t} + \sigma_{2G}dW_{2t}] + \mathscr{W}_t^*\frac{1}{\gamma}[\lambda_1 dW_{1t} + \lambda_2 dW_{2t}] \\ &= \text{drift} + \left[\sigma_{1G} + \frac{1}{\gamma}\lambda_1\right]\mathscr{W}_t^* dW_{1t} + \left[\sigma_{2G} + \frac{1}{\gamma}\lambda_2\right]\mathscr{W}_t^* dW_{2t}, \end{aligned} \tag{26}$$

where the first equality uses the product rule, the second equality makes use of (19) and the chain rule, the third equality makes use of (18) and the definition of the Radon–Nikodym derivative (7), and the fourth equality makes use of (18) again. Comparing (26) with the self-financing condition, here written in component form,

$$\begin{aligned} d\mathscr{W}_t^* &= [r + \pi_{S,t}^*\sqrt{V_t}\lambda_1(V_t) + \pi_{D,t}^*(\sigma_{1D}\lambda_1(V_t) + \sigma_{2D}\lambda_2(V_t))]\mathscr{W}_t^* dt \\ &\quad - c_t dt + [\pi_{S,t}^*\sqrt{V_t} + \pi_{D,t}^*\sigma_{1D}]\mathscr{W}_t^* dW_{1t} + \pi_{D,t}^*\sigma_{2D}\mathscr{W}_t^* dW_{2t}, \end{aligned} \tag{27}$$

we see that we have established two simultaneous equations from which π^*_{St} and π^*_{Dt} can be determined

$$\pi^*_{S,t}\sqrt{v} + \pi^*_{D,t}\sigma_{1D} = \sigma_{1G} + \frac{1}{\gamma}\lambda_1(v), \quad \text{and} \quad \pi^*_{D,t}\sigma_{2D} = \sigma_{2G} + \frac{1}{\gamma}\lambda_2(v).$$

Solving these we ultimately arrive at

$$\pi^*_{St} = \frac{1}{\sqrt{v}}\left\{\frac{\lambda_1(v)}{\gamma} - \frac{\sigma_{1D}}{\sigma_{2D}}\frac{\lambda_2(v)}{\gamma} + \sigma_{1H} - \frac{\sigma_{1D}\sigma_{2G}}{\sigma_{2D}}\right\}, \tag{28a}$$

$$\pi^*_{Dt} = \frac{1}{\sigma_{2D}}\left\{\frac{\lambda_2(v)}{\gamma} + \sigma_{2G}\right\}. \tag{28b}$$

3.4. Theoretical summary

For the reader's convenience we here summarize the key results presented in the preceding sections.

Theorem 1. *Consider the control problem stated in* (11). *Then the optimal consumption is given by*

$$c^*_t = \alpha^{1/\gamma}\frac{\mathscr{W}^*_t}{G_t}. \tag{29}$$

Whilst the optimal portfolio weights are given by

$$\pi^*_{St} = \frac{1}{\sqrt{v}}\left\{\frac{\lambda_1(v)}{\gamma} - \frac{\sigma_{1D}}{\sigma_{2D}}\frac{\lambda_2(v)}{\gamma} + \sigma_{1G} - \frac{\sigma_{1D}\sigma_{2G}}{\sigma_{2D}}\right\}, \tag{30a}$$

$$\pi^*_{Dt} = \frac{1}{\sigma_{2D}}\left\{\frac{\lambda_2(v)}{\gamma} + \sigma_{2G}\right\}, \tag{30b}$$

which jointly lead to the optimal wealth process

$$\mathscr{W}^*_t = \frac{w_0 e^{-qt}\xi_t^{-1/\gamma}G_t}{G_0}. \tag{31}$$

Here $G_t = G(t, v)$ is the function

$$G_t = \alpha^{1/\gamma}\int_t^T e^{-(r+q)(s-t)}H^s_t ds + (1-\alpha)^{1/\gamma}e^{-(r+q)(T-t)}H^T_t, \tag{32}$$

where $H^s_t = H^s(t, v)$ is governed by the linear PDE

$$0 = \partial_t H^s + \alpha^0(t, v)\partial_v H^s + \frac{1}{2}\beta^2(t, v)\partial^2_{vv}H^s + \frac{1-\gamma}{2\gamma^2}||\boldsymbol{\lambda}(v)||^2 H^s, \tag{33}$$

with terminal condition $H_s^s = 1$. *Finally,* σ_{1D} *and* σ_{2D} *are the derivative diffusion coefficients established in* (4), σ_{1G} *and* σ_{2G} *are the diffusion coefficients of* dG_t *as established in* (23), *which in turn depend on the H diffusion coefficients* σ_{1H} *and* σ_{2H} *established in* (20).

Of particular interest are the cases where the relative importance weight, α, assumes, respectively, its maximum and minimum vales, $\{1, 0\}$.

Corollary 1. *For the investor who solely cares about optimizing her continuous rate of consumption process, the optimal policy is*

$$c_t^* = \frac{\mathscr{W}_t^*}{\int_t^T e^{-(r+q)(s-t)} H_t^s ds}, \tag{34}$$

where the optimal wealth process is given by

$$\mathscr{W}_t^* = \frac{w_0 e^{rt} \xi_t^{-1/\gamma} \int_t^T e^{-(r+q)s} H_t^s ds}{\int_0^T e^{-(r+q)s} H_0^s ds}. \tag{35}$$

At the other extreme, for the investor who solely seeks to maximize her bequest, the optimal portfolio policies are

$$\pi_{St}^* = \frac{1}{\sqrt{v}} \left\{ \frac{\lambda_1(v)}{\gamma} - \frac{1}{\sqrt{1-\rho^2}} \left[\rho + \frac{\sqrt{v} s \partial_s D}{\beta(t,v) \partial_v D} \right] \frac{\lambda_2(v)}{\gamma} - \sqrt{v} s \frac{\partial_s D}{\partial_v D} \frac{\partial_v H_t^T}{H_t^T} \right\}, \tag{36a}$$

$$\pi_{Dt}^* = \frac{D \lambda_2(v)}{\sqrt{1-\rho^2} \beta(t,v) \partial_v D} + \frac{D}{\partial_v D} \frac{\partial_v H_t^T}{H_t^T}, \tag{36b}$$

with optimal wealth process

$$\mathscr{W}_t^* = \frac{w_0 e^{rt} \xi_t^{-1/\gamma} H_t^T}{H_0^T}. \tag{37}$$

We shall have much more to say on this matter below, where we investigate the empirical performance of a portfolio manager faced with stochastic volatility and the intention of wealth maximization.

4. Example: The Heston Model

Undoubtedly, the most well-known of all stochastic volatility models is that proposed by Heston (1993). This model stands out for a number of reasons: first, the variance process is non-negative and mean-reverting, which harmonizes with market data; secondly, the model is sufficiently parsimonious to allow for swift calibrations (Ribeiro and Poulsen, 2013 and Weron and Wystup, 2011); third, as exposed below, it famously admits comparatively simple expressions for plain vanilla options; finally, said expressions yield implied volatilities which are found to fit the empirically observed volatility smile closely for a broad range[7] of medium-seized times to maturity (Weron and Wystup, 2011).

Formally, the Heston model is the model by Cox, Ingersoll, and Ross (1985) for the variance process,

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v\sqrt{V_t}(\rho dW_{1t} + \sqrt{1-\rho^2}dW_{2t}), \tag{38}$$

where κ, θ, and σ_v are non-negative parameters which signify the speed of mean reversion, the long term variance, and the volatility of variance, respectively. Insofar as the Feller condition is satisfied, it can be shown that the variance process stays strictly positive at all times.[8] Moreover, the distribution of V_t under (38) is non-central chi-squared, which in the asymptotic limit $t \to \infty$ tends towards a gamma distribution. This effectively disposes of one of the key shortfalls of classical GBM valuation as the resulting density function of log returns will be fatter than the bell curve.

4.1. Vanilla valuation

What really propelled the Heston model into the academic limelight is largely its ability to price European calls (and *ipso facto* European puts). For the reader's convenience, we here briefly review the valuation formula, and tie it to the theory of pricing in incomplete markets alluded to in Sec. 2.1. Specifically, the relevant pricing PDE (8) is of the form

$$\begin{aligned}0 = {}& \partial_t D + rs\partial_s D + \{\kappa(\theta - v) - \sigma_v\sqrt{v}[\rho\lambda_1(v) + \sqrt{1-\rho^2}\lambda_2(v)]\}\partial_v D \\ &+ \frac{1}{2}vs^2\partial_{ss}^2 D + \frac{1}{2}\sigma_v^2 v\partial_{vv}^2 D + \rho\sigma_v vs\partial_{sv}^2 D - rD,\end{aligned} \tag{39}$$

subject to $D(T', s) = [\phi(s-K)]^+$, where ϕ is a binary variable which takes on the value $+1$ if the option is a call, and -1 if the option is a put. Upon solving this equation, Heston crucially makes the assumption that the *market price of volatility*

[7]Matching the smile for very short or very long times to maturity proves more difficult. In particular, with regards to the former, the so-called volatility of variance, σ_v, tends to explode, which indicates that there is a jump effect neglected by the dynamics.

[8]It is dubious that calibrated parameters actually satisfy this condition; Ribeiro and Poulsen (2013).

risk, λ_v, here defined as[9]

$$\lambda_v \equiv \sigma_v[\rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2], \tag{40}$$

is proportional to $\sqrt{v}$, i.e.,

$$\exists\bar{\lambda}_v \in \mathbb{R} \text{ s.t. } \lambda_v(v) = \bar{\lambda}_v\sqrt{v}. \tag{41}$$

We call this the Heston assumption and note that it constitutes a concrete instantiation of the weak Heston assumption explicated above. Nonetheless, based on our desire to solve the PDE for the H-function, it will in fact be convenient to assume something slightly stronger, viz.

Assumption 2. There exist constants $\bar{\lambda}_1$ and $\bar{\lambda}_2$ such that $\lambda_1(v) = \bar{\lambda}_1\sqrt{v}$ and $\lambda_2(v) = \bar{\lambda}_2\sqrt{v}$. We call this the strong Heston assumption.

Remark 1. A partial motivation for (41) is provided through Breeden's consumption-based model, $\lambda_v(V_t)dt = \gamma\mathbb{C}\text{ov}[dV_t, dc_t/c_t]$, when the consumption process is chosen as in the (general equilibrium) Cox–Ingersoll–Ross model (see Heston (1993)). Less generously, we might view it as a postulate detached from empirical evidence, which purposefully has been engineered in order to allow (39) to be solved. Be that as it may, under the proportionality assumption it can be shown that the ELMM, $\mathbb{Q}$, exists and that discounted asset prices are true Martingales in so far as certain inequalities on the parameters are satisfied cf. Wong and Heyde. This may be taken as a formal justification for Heston's well-known valuation formula.

Theorem 2 (Heston's valuation formula for European vanillas). *The no-arbitrage price of a European vanilla option with maturity time T' is given by*

$$\begin{aligned} D(t,s,v) &= \text{HestonVanilla}(\kappa,\theta,\sigma_v,\rho,\bar{\lambda}_1,\bar{\lambda}_2,r,v,s,K,\tau',\phi) \\ &= \phi\{sQ_1(\phi) - e^{-r\tau'}KQ_2(\phi)\}, \end{aligned} \tag{42}$$

where $\phi = +1$ if D is a call, and $\phi = -1$ if D is a put, $\tau' \equiv T' - t$ is the time to maturity,

$$Q_j(\phi) \equiv \frac{1-\phi}{2} + \phi P_j(\ln s, v, \tau', \ln K), \tag{43}$$

[9]The market price of volatilty risk (40) is a concept which arises naturally insofar as the dynamical equations (2) have *not* had their random components decorellated through a Cholesky decomposition. Specifically, for the market price of risk vector, $\boldsymbol{\lambda} = \boldsymbol{\sigma}^{-1}$ (excess return vector), we would set $\boldsymbol{\sigma} = [\sqrt{V_t}, 0; D^{-1}s\sqrt{v}\partial_s D + D^{-1}\rho\sigma_v\sqrt{v}\partial_v D, D^{-1}\sqrt{1-\rho^2}\sigma_v\sqrt{v}\partial_v D]$, whilst Heston sets $\boldsymbol{\sigma} = \boldsymbol{\sigma}' := [\sqrt{V_t}, 0; D^{-1}\sqrt{v}\partial_s D, D^{-1}\sigma_v\sqrt{v}\partial_v D]$ (the latter is related to the former through the multiplication of the lower triangular matrix $\mathbf{L} = [1, 0; \rho, \sqrt{1-\rho^2}]$: $\boldsymbol{\sigma} = \boldsymbol{\sigma}'\mathbf{L}$). For convenience, Heston also absorbs the constant σ_v in his definition.

is a probability weight for $j = 1, 2,$ *and we have defined the functions*

$$P_j(\ln s, v, \tau', \ln K) \equiv \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re\left\{ \frac{e^{-i\varphi \ln K} f_j(\ln s, v, \tau', \varphi)}{i\varphi} \right\} d\varphi, \tag{44a}$$

$$f_j(\ln s, v, \tau', \varphi) \equiv \exp\{C_j(\tau', \varphi) + D_j(\tau', \varphi)v + i\varphi \ln s\}, \tag{44b}$$

$$D_j(\tau', \varphi) \equiv \frac{b_j - \rho\sigma_v \varphi i + d_j}{\sigma_v^2} \left(\frac{1 - e^{d_j \tau'}}{1 - g_j e^{d_j \tau'}} \right), \tag{44c}$$

$$C_j(\tau', \varphi) \equiv r\varphi i \tau' + \frac{a}{\sigma_v^2} \left\{ (b_j - \rho\sigma_v \varphi i + d_j)\tau' - 2\ln\left(\frac{1 - g_j e^{d_j \tau'}}{1 - g_j} \right) \right\}, \tag{44d}$$

where $d_j \equiv ((\rho\sigma_v\varphi i - b_j)^2 - \sigma_v^2(2u_j\varphi i - \varphi^2))^{1/2}$, $g_j \equiv (b_j - \rho\sigma_v\varphi i + d_j)/(b_j - \rho\sigma_v\varphi i - d_j)$, $a \equiv \kappa\theta$, $u_1 \equiv \frac{1}{2}$, $u_2 \equiv -\frac{1}{2}$, $b_1 \equiv \kappa + \bar{\lambda}_v - \rho\sigma_v$, *and* $b_2 \equiv \kappa + \bar{\lambda}_v$.

Remark 2. The formula stated in (42) is rather unconventionally expressed in terms of the market price of risk constants $\bar{\lambda}_i$, $i = 1, 2$ along with the $\mathbb{P}$-parameters of the variance process. The reason, as will be highlighted in the following section and by the estimation procedure employed for the empirical study, is that the trading strategies are explicitly dependent on the statistical parameters and market prices of risk separately. More commonly, the valuation formula is specified directly in terms of the risk-neutral $\mathbb{Q}$-parameters where the market prices of risk are implicit: $\kappa^{\mathbb{Q}} \equiv \kappa + \sigma_v(\rho\bar{\lambda}_1 + \sqrt{1-\rho^2}\bar{\lambda}_2)$ and $\theta^{\mathbb{Q}} \equiv \theta\kappa/\kappa^{\mathbb{Q}}$, while the diffusion parameters are invariant.

4.2. The optimal Heston controls

The time is now ripe to talk about the optimal consumption-investment policies for a utility maximizer embedded in a Hestonian economy. For simplicity, we here restrict out attention to the specific cases of $\alpha = 1$ and $\alpha = 0$, although the general cases are straightforwardly obtained from Theorem 1. Specifically, from the generic optimal control functions declared in Corollary 1, it follows that we must determine $\sigma_{1D}, \sigma_{2D}, \sigma_{1H^s},$ and σ_{2H^s} and thence the quantities $\partial_s D, \partial_v D, H^s,$ and $\partial_v H^s$. We do this over the two subsequent lemmas.

Lemma 1. *The option delta is given by*

$$\Delta_t \equiv \partial_s D = \phi Q_1(\phi), \tag{45}$$

while the option vega is given by

$$\nu_t \equiv \partial_v D = s\nu_1(\ln s, v, \tau', \ln K) - Ke^{-r\tau'}\nu_2(\ln s, v, \tau', \ln K), \tag{46}$$

where

$$\nu_j(\ln s, v, \tau', \ln K) \equiv \frac{1}{\pi}\int_0^\infty \Re\left\{\frac{D_j(\tau',\varphi)e^{-i\varphi \ln K} f_j(\ln s, v, \tau', \varphi)}{i\varphi}\right\} d\varphi. \tag{47}$$

Proof. See the appendix. □

Lemma 2. *Suppose a relative risk aversion* $\gamma > 1$ *and that the strong Heston assumption holds, then the* H^s *function is exponential affine. Specifically, for* $s \in [t, T]$

$$H^s(t, v) = \exp\{A^s(\tau) + B^s(\tau)v\}, \tag{48}$$

where $\tau \equiv s - t$. *Here* $B^s : [0, s] \mapsto \mathbb{R}$ *is the monotonically decreasing function*

$$B^s(\tau) = \frac{1-\gamma}{\gamma^2}(\bar{\lambda}_1^2 + \bar{\lambda}_2^2) \cdot \frac{e^{\omega\tau} - 1}{(\omega + \alpha)(e^{\omega\tau} - 1) + 2\omega}, \tag{49}$$

where $\alpha \equiv \kappa + (1 - 1/\gamma)\bar{\lambda}_v$ *and*

$$\omega \equiv \sqrt{\alpha^2 + \sigma_v^2 \frac{\gamma - 1}{\gamma^2}(\bar{\lambda}_1^2 + \bar{\lambda}_2^2)},$$

while $A^s : [0, s] \mapsto \mathbb{R}$ *is the monotonically increasing function*

$$A^s(\tau) = \frac{\kappa\theta}{\alpha^2 - \omega^2}\left\{(\alpha + \omega)\tau + 2\ln\left|\frac{2\omega}{(\alpha + \omega)(e^{\omega\tau} - 1) + 2\omega}\right|\right\}. \tag{50}$$

Proof. See the appendix. □

Putting these results together we arrive at the following theorem.

Theorem 3. *For the investor who solely cares about optimizing her continuous rate of consumption process in a Hestonian economy, the optimal policy is*

$$c_t^* = \frac{w_0 e^{-qt}\xi_t^{-1/\gamma}}{\int_0^T e^{-(r+q)s + A^s(s) + B^s(s)v} ds}. \tag{51}$$

At the other extreme, for the investor who solely seeks to maximize her bequest in a Hestonian economy, the optimal portfolio policies are

$$\pi_{S,t}^* = \frac{\bar{\lambda}_1}{\gamma} - \frac{1}{\sqrt{1-\rho^2}}\left[\rho + \frac{s}{\sigma_v}\frac{\Delta_t}{\nu_t}\right]\frac{\bar{\lambda}_2}{\gamma} - s\frac{\Delta_t}{\nu_t}B^T(\tau), \tag{52}$$

while the optimal vanilla option weight is

$$\pi_{D,t}^* = \frac{D_t\bar{\lambda}_2}{\gamma\sigma_v\sqrt{1-\rho^2}\nu_t} + \frac{D_t}{\nu_t}B^T(\tau), \tag{53}$$

where B is defined in (49), D_t is the option price given by (42), Δ_t is the option delta given in (45), and ν_t is the option vega given in (46). Note that the time parameter in B is $\tau = T - t$ (the investment horizon), while for option quantities $\{D, \Delta, \nu\}$ it is $\tau' = T' - t$ (the time-to-maturity of the option).

Remark 3. We note that the first term in (52) is Merton's optimal stock weight in a simple (B, S)-economy with constant volatility. More generally, referencing standard results in the literature,[10] we see that the first two terms in (52) and the first term in (53) constitute the optimal portfolio weights in a (B, S, D)-economy for a utility maximizing investor who disregards stochastic fluctuations in the state variable v (otherwise known as the *myopic* or *1-period* strategy). Thus, the hedge against stochastic volatility is nested in the time-price-volatility dependent term $-s\frac{\Delta_t}{\nu_t}B(\tau)$ in (52) and $\frac{D_t}{\nu_t}B(\tau)$ in (53). In this connection, we note that Δ is a function bounded by the interval [0, 1] for a call option ($[-1, 0]$ for a put option), whilst $B(\tau)$ is bounded by the interval $((1-\gamma)(\bar{\lambda}_1^2 + \bar{\lambda}_2^2)/(\gamma^2[\omega + \alpha]), 0]$. The quantities $s, D,$ and ν are all positive and unbounded from above. The signs of the volatility hedge corrections on the stock and the derivative are thus, respectively, positive and negative if D is a call option, and negative and negative if D is a put option.

To appreciate the implications of Theorem 3, Fig. 1 plots: (i) the optimal consumption process at different time to maturity, (ii) the optimal (bank, stock, ATM call option)-weights for different times to maturity. In both cases, (s, v) is held constant at $(100, 0.030)$. We assume that $r = 0.02$, $\delta = 0.015$, $T = T' = 1$, $\gamma = 2$, and $w_0 = 1000$. Other parameters are estimated from the S&P500 index as exhibited in Table 1.

Upon examining Fig. 1, we make the following observations: for the consumption process, there is an exponential increase in the rate at which the agent consumes, which explodes at $\tau \to 0$. This is to be expected: as shorter the time horizon, the faster the agent desired to burn through her endowment. For comparative purposes the corresponding consumption strategy for a Merton economy (constant volatility) has also been plotted. Interestingly, the difference between the two plots is very hard to discern.

As for the bequest maximizing agent, the difference between the Merton realm and the Heston realm is more noticeable — at least when one allows for derivative trading. Specifically, while Liu's portfolio weight only adds marginally to the optimal Merton weight on the stock, we see that having access to derivative trading prompts the investor to drastically increase her holding in the stock, by decreasing her long position in the money account and shorting the call option at a rather

[10]See, for example, (Munk (2013), Theorems 6.2 and 7.5).

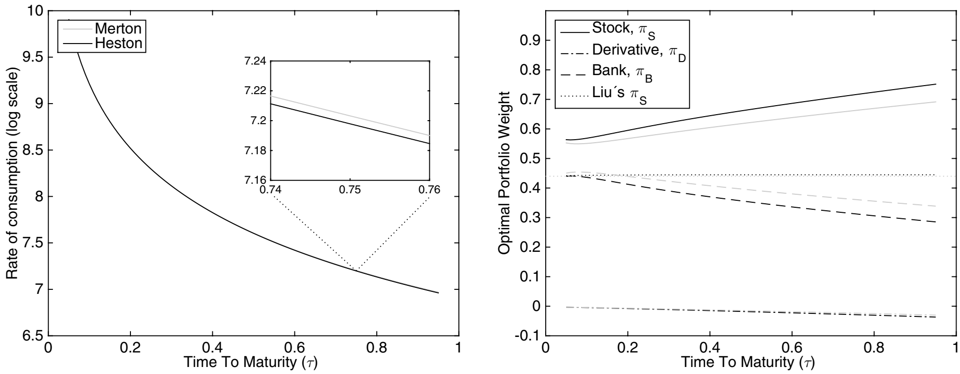

Fig. 1. Left: The optimal rate of consumption strategy for various times to maturity with s and v held constant when $\alpha = 1$. For clarity, the y-axis is in log-scale. On the same plot, the analogous consumption strategy for an agent consuming in a Mertonian economy (constant variance) has been plotted. Evidently, the difference is miniscule. Right: The optimal investment strategies in a (bank, stock, call option)-economy for different times to maturity with s and v held constant when $\alpha = 0$. Note that the investor shorts the derivative in order to enter into a significant long position in the underlying stock and deposit money in the bank. The dotted line shows Liu's optimal portfolio weight on the stock when the investor disregards derivatives. The grey "shadow" lines represent the corresponding strategies when we do not hedge stochastic variations in the state variable (volatility). For Liu's model, this corresponds to the Merton weight λ_1/γ.

modest level (this makes good sense: by shorting the call, the investor has a *negative* exposure to the risk endemic to the variance process thereby collecting positive risk premium, (Munk, 2013)). Indeed, there seems to be a visually noticeable difference between merely adding a second risky asset to the portfolio (grey lines) and adding that risky asset, and going through the trouble of hedging the stochastic fluctuations in the underlying state variable (black lines) (which is what we have done in the analysis above). One would therefore imagine that the volatility hedge corrections *seemingly* have the magnitude to perturb the terminal wealth of a rational investor by a measurable amount. Yet, this is in fact *not* the case when we Monte Carlo simulate the wealth process of an investor trading in a Hestonian economy: although the expected return is higher for someone who hedges volatility vis-à-vis one who does not, so is the associated variance.

Table 1. Estimated model parameters with the two-step approach. The estimates are based on 3,909 daily observations of the stock price, the variance, and the call price. Note that $\bar{\lambda}_1 > 0$ whilst $\bar{\lambda}_2 < 0$.

	κ	θ	σ_v	ρ	$\bar{\lambda}_1$	$\bar{\lambda}_2$
Estimate	9.71	0.030	2.72	−0.17	0.88	−0.29

A Welch's t-test therefore cannot reject the null hypothesis that the two trading strategies have equal returns (p-value ≈ 0.65). This suggests that the real capital gains (if any) are to be garnered from access to the derivative security, and not the hedge corrections to volatility per se, in accordance with the findings by Liu and Pan (2003).

4.3. The value of derivative trading

Whilst the inclusion of a derivative to the investment portfolio has a significant impact on the optimal trading strategy, one should be more vigilant of a potential *value-gain* from such a position. In particular, since both $(\pi_B^*, \pi_S^*, \pi_D^*)$ and Liu's strategy $(\pi_B^{\text{Liu}}, \pi_S^{\text{Liu}})$ are optimal in the sense that they maximize the expected utility of terminal wealth, their respective utilities are the quantities to consider for a comparison. In fact, Liu's optimum is a constrained strategy of the former, in that no derivative trading is allowed ($\pi_D \equiv 0$) and thus suboptimal: with $J(w_0, \pi) = \mathbb{E}[e^{-\delta T} u(\mathscr{W}_T)]$ denoting the expected utility from investing an initial wealth w_0 according to a strategy $\boldsymbol{\pi}$, we readily have that

$$J(w_0, \boldsymbol{\pi}^{\text{Liu}}) \leq J(w_0, \boldsymbol{\pi}^*) = \sup_{\boldsymbol{\pi}} J(w_0, \boldsymbol{\pi}).$$

A common measure for the comparison of two such strategies in monetary terms is the *certainty equivalent in wealth*

$$\bar{w}^* = \sup\{\bar{w} \geq 0 : J(w_0, \boldsymbol{\pi}^{\text{Liu}}) \leq J(w_0 - \bar{w}, \boldsymbol{\pi}^*)\},$$

which gives the reduction in initial wealth the investor is willing to sacrifice in order to trade in the derivative market (see (Munk (2013), Chap. 5.4)). For the optimal strategy, we have an optimal terminal wealth $\mathscr{W}_T^*$ from (37) that yields

$$J(w_0, \boldsymbol{\pi}^*) = e^{(r(1-\gamma)-\delta)T} \frac{w_0^{1-\gamma}}{1-\gamma} H(0, v)^{\gamma}.$$

Similarly, following Ellersgaard and Tegnér (2015), the optimal expected utility from investing in a stock-bond economy, $J(w_0, \boldsymbol{\pi}^{\text{Liu}})$, is identical, albeit with different exponential coefficients, $\hat{A}$ and $\hat{B}$, for the H function.[11] A straightforward calculation now gives the certainty equivalent

$$\bar{w}^* = (1 - [e^{\hat{A}(T) - A(T) - (\hat{B}(T) - B(T))v}]^{\frac{\gamma}{1-\gamma}}) w_0. \tag{54}$$

Figure 2 depicts certainty equivalents in wealth as a fraction of w_0 for different levels of risk aversion γ. The certainty equivalent is based on the estimated

[11]The functions $\hat{A}$, $\hat{B}$ are identical to (50) and (49) with slightly modified coefficients. For their exact expressions, we refer to Ellersgaard and Tegnér (2015).

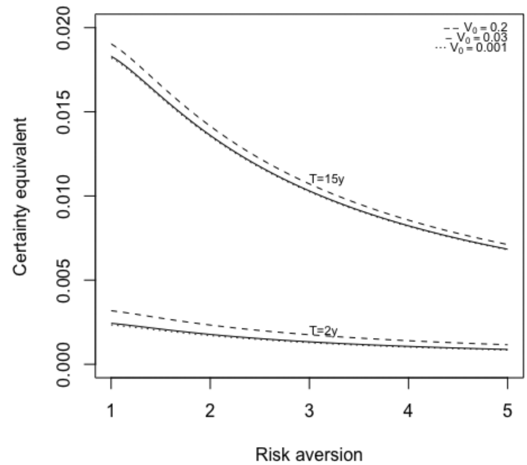

Fig. 2. Certainty equivalents in initial wealth: the fraction of wealth an investor is willing to sacrifice to be able to trade in the derivative market, as a function of investor risk-aversion. The upper curves show wealth equivalents for a trading horizon of 15 years while the lower curves represent a two-year trading horizon. Three different levels for the current volatility are included in the figure.

parameters from the S&P500 data (Table 1) and two different investment horizons (two years and 15 years) corresponding to the horizons that will be used for the empirical trading experiment. We have included three different levels of current volatility: $v = 0.03$ is equal to the estimated long-term variance θ, $v = 0.001$ is the same order of magnitude as the minimum historical S&P500 variance, while $v = 0.20$ could be considered to be a relatively high (and rare) level, cf. Fig. 3.

Clearly, Fig. 2 demonstrates that the investment horizon has a high impact on the certainty equivalent whilst the current variance level plays a minor role: the

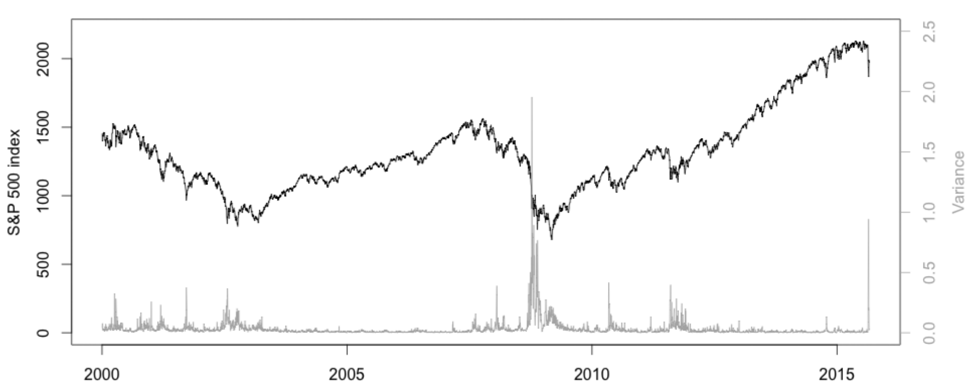

Fig. 3. Daily market prices and measured variance of the S&P500 index from the period 2000-01-03 to 2015-08-31. The price data is sourced from Wharton Research Data Services, while the variance data is sourced from the Oxford-Man Institute's realized library.

investor is prepared to refrain from a few percentage points from her initial wealth in return of being able to trade in the derivative market, and the longer the period, the more to gain from trading. The current variance is ephemeral, and thus yields a negligible effect. We further note that the wealth equivalent decreases with level of risk aversion γ: more risk-averse investors will be less prone towards risky trading of derivatives and thus have less to gain from this business.

For our empirical study, we assume $\gamma = 2$ which yields certainty equivalents $\bar{w}^* \approx 0.014 \cdot w_0$ (for initial variance levels between 0.001 and 0.20) for the first experiment with an investment horizon of $T = 15$ years and $\bar{w}^* \approx 0.0018 \cdot w_0$ for the second experiment with $T = 2$ years. For both horizons, only a very modest fraction of initial wealth has to be sacrificed for the access to derivative trading. Whether we will actually see any difference in *realized utility* (expressed in monetary terms with the realized certainty equivalent) will be revealed in the empirical trading experiment below.

Finally, note that the optimal expected utility is independent of which particular European option we consider to be constituent of the underlying financial market. Consequently, the implied expected certainty equivalent with respect to Liu's strategy is the same regardless of whether we compare trading in an economy with a call, put, or straddle, as will be the case in the forthcoming section.

5. The Empirical Perspective

Based on the optimal portfolio policies in a Heston-driven (B, S, D)-economy, we proceed to perform an empirical experiment which aims to measure the degree to which the inclusion of options impacts the financial wealth of a bequest-maximizing investor. In particular, we set out to perform an automated trading experiment, where we let historical market prices from the S&P500 index play the role of the fundamental risky security, and where the market prices of European options written on the same index constitute the derivative available in the economy. We use market interest rates for the money account.

5.1. Market data

For the tradable stock, we use 3,909 daily closing prices of the S&P500 index from the period 2000-01-03 to 2015-08-31. The market price, sourced from Wharton Research Data Services,[12] is plotted in Fig. 3 together with the daily variance. The variance process is measured from high-frequency quotes of the index ($\sim$1 min) with the realized volatility measure and we use the precomputed estimates from

[12] https://wrds-web.wharton.upenn.edu/wrds/.

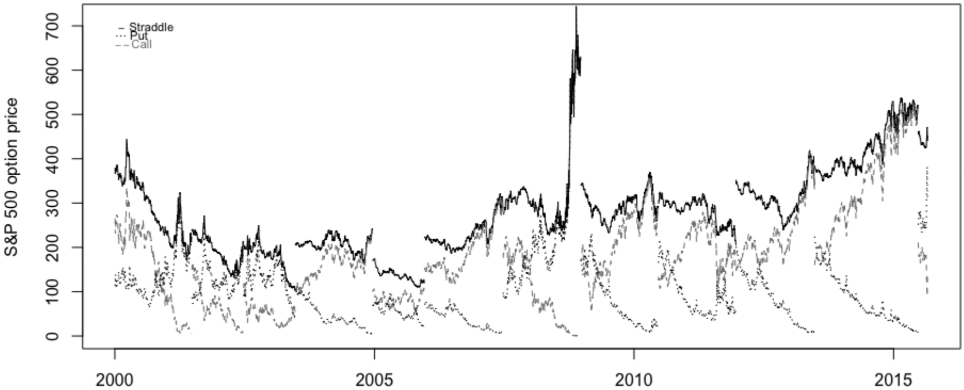

Fig. 4. Daily mid-market prices of European options on the S&P500 index from the period 2000-01-03 to 2015-08-31. Solid black line represents the straddle, dotted black line represents the put, and dashed grey line represents the call. The price discontinuities indicate a change in option strike and maturity according to Fig. 5. Consequently, there is no trading at these time-points.

the Oxford-Man Institute's realized library.[13] For details on the realized volatility measure, see, for instance, Andersen and Benzoni (2009).

For the tradable derivative, we consider the daily mid-market prices of the European call and put options written on the S&P500 index from the same time period. We use the Option Metrics database sourced through Wharton Research Data Services. Figure 4 shows the prices after converting to zero-dividend prices.[14] We also consider a combination of the put and call with identical strike-maturity specification for a straddle option. The time period covers prices of 3×12 options with medium-sized times to maturity starting at $\sim$24 months, while the strike-price of each option is selected to be at-the-money at initiation (or as close as possible thereto subject to data availability), thereby sowing the seed for high exposure to volatility risk. The strike-price and time-to-maturity structure is shown in Fig. 5.

For the risk-free asset in the (B,S,D) economy, we use the daily short-term LIBOR rate for an interest to the money account. The LIBOR market-data is obtained from the Option Metrics database as well.

[13]The Realized Library version 0.2 by Heber, Gerd, Lunde, Shephard, and Sheppard (2009). Available at http://realized.oxford-man.ox.ac.uk.

[14]The zero-dividend price is calculated by first reversing the Black–Scholes implied volatility with the effective dividend yield of the S&P500 index, and then recalculating the Black–Scholes price with the dividend set to zero.

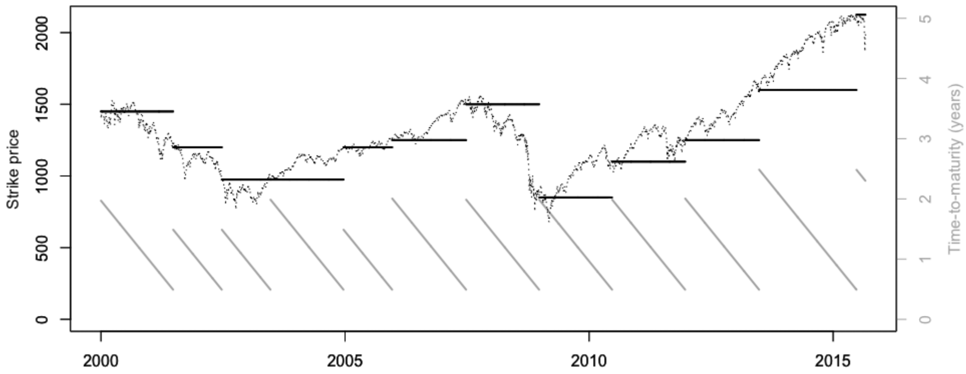

Fig. 5. The strike-price (solid black line) and time-to-maturity (solid grey line) for the options with prices shown in Fig. 4. The dotted line represents the S&P500 index level. Note the high level of moneyness (strike $\approx 2 \times$ spot) for the sixth option in time, which results in very high prices at the end of 2008 for the straddle and put (cf. Fig. 4).

5.2. Parameter estimation

In our empirical experiment, we will invest in the market according to the optimal portfolio weights $(\pi^*_{Bt}, \pi^*_{St}, \pi^*_{Dt})$ given by (52) and (53) which are functions of the model parameters under the real-world measure $\mathbb{P}$, the current stock price, the option price, and variance level at time t (along with the subjective risk-aversion parameter γ). To this end, we estimate the model parameters with the following approach: firstly, we estimate $(\kappa, \theta, \sigma_v, \rho)$ of the Cox–Ingersoll–Ross process from the daily variance data with a maximum likelihood method.[15] Secondly, based on these parameters we minimize the squared error between observed S&P500 call option prices $\{\hat{C}_t\}$ and the corresponding Heston prices $\{C^{He}_t = C^{He}_t(\bar{\lambda}_1, \bar{\lambda}_2)\}$, to determine the market prices of risk $(\bar{\lambda}_1, \bar{\lambda}_2)$. The strike-maturity structure is chosen as in Fig. 5. Specifically, we solve the minimization problem

$$(\bar{\lambda}_1, \bar{\lambda}_2) = \underset{(\bar{\lambda}_1, \bar{\lambda}_2)}{\text{argmin}} \sum_{t \in \mathbb{T}} (\hat{C}_t - C^{He}_t(\bar{\lambda}_1, \bar{\lambda}_2))^2,$$

where $\mathbb{T} = \{$trading days between 2000-01-03 and 2015-08-31$\}$. The resulting parameters estimated from the market data are given in Table 1.

A few remarks on the above estimation procedure are in order here. First, note that when we will use the parameter estimates for the forthcoming trading experiment, we employ ex-ante estimates based on the actual "future" market data. An alternative is to estimate parameters from the historical data prior to the trading

[15]We use numerical optimization of a Gaussian likelihood with exact moments, from the method of Sørensen (1999) based on estimating functions.

period. However, due to the amount of available data, this would impair the accuracy our estimates. Since we are primarily interested in the efficiency of the trading strategy, (and not in the parameter estimation problem per se) we require as many robust estimates as possible. Thus, we opt for the former alternative. Secondly, we use a rather unconventional estimation procedure whereby we separately estimate the volatility parameters under the statistical measure $\mathbb{P}$, followed by a mean-square optimization to back out the market prices of risk. A more commonplace approach is to formulate the pricing model under $\mathbb{Q}$ directly — see Remark 2 — and to estimate the risk-neutral parameters alone from the option data through mean square principles. This approach is commonly referred to as model-to-market calibration, and typically a whole surface of option prices is employed for the day-to-day estimation of the parameters. The reason for not using a calibration method is that we require the statistical CIR-parameters for the optimal strategies as well as the market prices of risk separately. This in contrast to calibrated risk-neutral parameters, which have the market prices of risk implicitly embedded in their drift coefficients. Our focus is the estimation of the real-world dynamics of volatility which best serves the dynamical trading experiment. Finally, the mean-reversion speed κ and vol-of-vol σ_v recorded in Table 1 are estimated to be relatively high (Liu and Pan (2003) report $\kappa = 5$ and $\sigma_v = 0.25$ as their empirical calibrates). One explanation for this is that we use the daily data for the experiment, which results in a rather "spiky" realized variance of the S&P500 index, cf. Fig. 3. Consequently, the market prices of risk are estimated at relatively low values, which is reflected in the expected excess return of the index: $\mu_S(t, v) - r = \bar{\lambda}_1 v \approx \bar{\lambda}_1 \theta = 2.64\%$, whilst Liu and Pan set $\bar{\lambda}_1 = 4$ to obtain an expected premium of $\bar{\lambda}_1 \theta = 4 \cdot 1.69\% = 6.76\%$. Also, note that the market price of volatility risk $\bar{\lambda}_2$ associated with W_2 is negative, corroborating standard empirical findings a la Bakshi and Kapadia (2003).

5.3. Empirical trading experiment

With market data from the S&P500 index and the LIBOR rate, we set out to perform an empirical trading experiment. We intend to invest in a portfolio according to the optimal strategy $(\pi^*_{Bt}, \pi^*_{St}, \pi^*_{Dt})$ and we trade dynamically with daily rebalancing as the time evolves during the period 2000-01-03 to 2015-08-31. Hence, if ΔX_{t_i} denotes the daily price-change from time t_i to t_{i+1} of an asset in the economy, this means that we realize a daily change in the portfolio value

$$\Delta \mathscr{W}_{t_i} = \mathscr{W}_{t_i} \left(\pi^*_{Bt_i} \frac{\Delta B_{t_i}}{B_{t_i}} + \pi^*_{St_i} \frac{\Delta S_{t_i}}{S_{t_i}} + \pi^*_{Dt_i} \frac{\Delta D_{t_i}}{D_{t_i}} \right),$$

where $\mathscr{W}_{t_0} = w_0$. Here D represents the price of the derivative from the set {call, put, straddle}, the latter of which is the option strategy formed by combining a put and a call with identical strike-maturity structures.[16] Note that the portfolio is self-financing: an initial amount w_0 is invested at the initial time and there is no infusion or withdrawal of capital from the portfolio during the investment period.

In addition to the parameters in Table 1, we fix the constant of relative risk aversion to be $\gamma = 2$. For comparison purposes, we include a "naive" trading strategy with a constant equal weight invested in each asset, $(\pi_{Bt}, \pi_{St}, \pi_{Dt}) = (1/3, 1/3, 1/3)$. We also trade according to Liu's optimal investment strategy in the limited economy, that is, we invest in the stock and the bond with portfolio weights $(\pi_{Bt}^{\mathrm{Liu}}, \pi_{St}^{\mathrm{Liu}})$ while $\pi_{Dt} \equiv 0$ for the option. With this in mind, we conduct the following two experiments for each of the trading strategies:

(1) **Trading throughout 2000–2015.** We set the investment period to be 2000-01-03 to 2015-08-31 and initialize the portfolios with a wealth $w_0 = 1{,}000$. We do not trade over the dates when a "new" option is introduced, i.e., every time there is a change in the expiry date, since this would give false price-moves of the option due to changes in the strike price and expiry (cf. Figs. 4 and 5). The same rule pertains to Liu's strategy, even though there is no trading in the option.

The realized wealth paths from trading according to the strategies are shown in Fig. 6 where, respectively, the call, the put, and the straddle are representatives of the derivative in the economy. The greatest terminal wealth $\mathscr{W}_T = 1{,}662$ is realized with the optimal strategy in the case when the call option represents the underlying derivative. Trading optimally in the straddle yields a terminal wealth of 1,563 while the put yields 1,459. Liu's strategy terminates at 1,522 and for comparison, the naive-straddle strategy comes out with the least: 1,284.

For the optimal strategies to be equivalent with Liu's trading outcome, we have to initiate the optimal call-portfolio with at least $w_0 = 918$, which gives a terminal wealth of 1,522.[17] Since our experiment results in a single wealth-path here, this yields equivalent realized utilities as well. Hence, in terms of a relative certainty equivalent, a reduction of 8.1% from the initial wealth of the optimal call-trading strategy makes it comparable to the strategy being

[16]The optimal weights given in Eqs. (52) and (53) are calculated with the option delta and vega. Since $\Delta^{\mathrm{put}} = \Delta^{\mathrm{call}} - 1$ by the put-call-parity, $\Delta^{\mathrm{straddle}} = 2\Delta^{\mathrm{call}} - 1$ for the straddle. Similarly, $\nu^{\mathrm{call}} = \nu^{\mathrm{put}}$ such that $\nu^{\mathrm{straddle}} = 2\nu^{\mathrm{call}}$. Consequently $\pi^*_{\mathrm{straddle}} = \frac{1}{2}(\pi^*_{\mathrm{call}} + \pi^*_{\mathrm{put}})$ and for the optimal weight in the stock $\pi^*_{\mathrm{S,straddle}} = \frac{1}{2}(\pi^*_{\mathrm{S,call}} + \pi^*_{\mathrm{S,put}})$.

[17]The initial wealth w_0 which yields a realized terminal wealth (realized utility) equal to that of Liu's strategy is found with the bisection method with a tolerance of $< 0.01\%$.

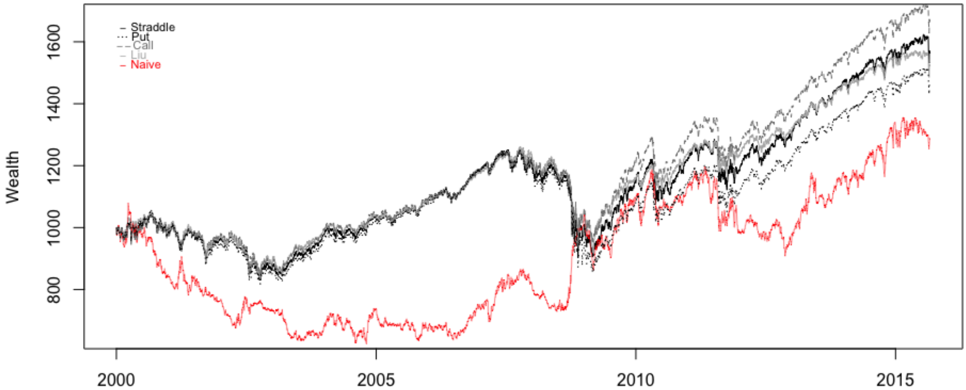

Fig. 6. (Color online) Trading throughout 2000-01-03 to 2015-08-31: wealth processes from trading in {LIBOR, S&P500, Option} according to the optimal investment strategies: dashed-grey for the call, dotted-black for the put, and solid-black for the straddle. Trading in {LIBOR, S&P500} with Liu's optimal strategy is plotted in solid-gray, while the naive strategy is represented in red.

restricted to bond and stock trading. As predicted by the certainty equivalent from the expected utility, $\bar{w}^* = 1.4\%$, this is a *notable* reduction which suggests that the two strategies differs in realized monetary performance. Similarly, the optimal strategy when trading in the straddle amounts to a realized certainty equivalent of 2.6%, while trading optimally in the put requires an increase of the initial wealth with certainty equivalent -4.3%.

Moreover, while we emphasize that our business manifestly is not Sharpe ratio optimization, it is arguably of some interest to see how the strategies compare in this regard. Per definition, the Sharpe ratio codifies the risk adjusted return delivered by a strategy, i.e., $\mathrm{Mean}(R_{t_i} - r_{t_i})/\mathrm{SD}(R_{t_i})$, where $R_{t_i} = \log(\mathscr{W}_{t_i}/\mathscr{W}_{t_{i-1}})$, is the daily returns of the investment portfolio and r_{t_i} is the daily return of the money account (the daily return from the LIBOR rate). The results are shown in Table 2. Strikingly, we here find that all strategies fall

Table 2. The Sharpe-ratio, mean, and standard deviation of daily portfolio returns from the strategies when trading throughout the whole period 2000-01-03 to 2015-08-31. The last column shows the annualized Sharpe-ratio. The daily mean-return of the money account is 0.0095%, which corresponds to an annualized return of 3.46%.

Strategy	Mean return	Std. Dev.	Sharpe	Sharpe annual
Optimal straddle	0.011%	0.65%	0.0029	0.055
Optimal call	0.013%	0.66%	0.0052	0.10
Optimal put	0.010%	0.68%	0.0018	0.034
Liu's strategy	0.010%	0.56%	0.0021	0.040
Naive	0.006%	0.88%	−0.0035	−0.068

short of having Sharpes statistically greater than zero (5% significance level) — indeed, at a similar level, there is no statistical basis for declaring that one Sharpe ratio is greater than the other (see Lo (2002), Mertens (2002), and Opdyke (2007) for the relevant test statistics).

(2) **Investment periods according to the option-expiry structure.** For our second empirical experiment, we reset our investment portfolio with an initial $w_0 = 1{,}000$ every time there is a new option, that is, every time there the expiry date and strike-price change (see Fig. 5). We set the investment period accordingly, i.e., to begin when we reset the portfolio and to end at the date on which we will reset the portfolio the next time. The resulting realized portfolio value-processes from the optimal (B, S, D) and (B, S) strategies are shown in Fig. 7.

For this experiment, we have a total of 12 investment periods with horizons around two years. Calculating the average realized utility as

$$\bar{J}(\pi) = \frac{1}{12} \sum_{j=1}^{12} \frac{(\mathscr{W}_{T_j})^{1-\gamma}}{1-\gamma},$$

where $\mathscr{W}_{T_j}$ is the realized terminal wealth of period j, we obtain $-9.636 \cdot 10^{-4}$ from the optimal strategy trading in the call option, $-9.696 \cdot 10^{-4}$ from the optimal strategy based on the straddle, $-9.70 \cdot 10^{-4}$ when trading according to Liu's strategy, and $-9.766 \cdot 10^{-4}$ from trading optimally in the put. The naive strategy trading in the straddle realizes the lowest utility of $-9.89 \cdot 10^{-4}$.

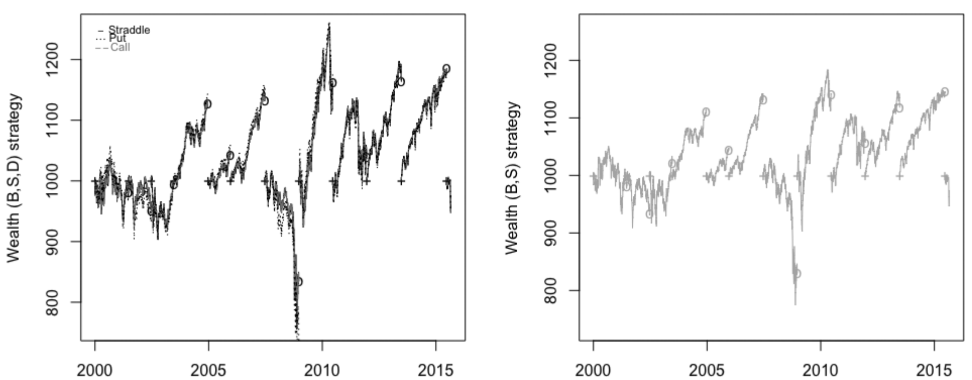

Fig. 7. Trading according to the option-expiry structure. Left: wealth process from trading in {LIBOR, S&P500, Option} according to the optimal investment strategies during investment periods that matches the expiry/strike structure of the options. A cross indicates the beginning of an investment period (with initial wealth $w_0 = 1{,}000$), while a circle shows the terminal wealth at the end of the period. Right: wealth process from trading in {LIBOR, S&P500} according to Liu's optimal investment strategy during the same investment periods.

Table 3. The Sharpe-ratio, mean, and standard deviation of daily portfolio returns from the strategies when trading according to the option-expiry structure. The last column shows the annualized Sharpe-ratio.

Strategy	Mean return	Std. Dev.	Sharpe	Sharpe annual
Optimal straddle	0.011%	0.65%	0.0030	0.058
Optimal call	0.013%	0.65%	0.0052	0.10
Optimal put	0.010%	0.67%	0.0005	0.010
Liu's strategy	0.011%	0.56%	0.0021	0.041
Naive	0.006%	0.88%	−0.0035	−0.068

In terms of a certainty equivalent, we have to initiate the optimal strategy trading in the call option (at the beginning of every period) with $w_0 = 993.5$ in order to obtain the same average realized utility as for Liu's strategy. This corresponds to a reduction of 0.65% of initial wealth, to be compared with the prediction in certainty equivalent $\bar{w}^* = 0.18\%$. Similarly, the optimal strategy trading in the straddle yields a realized equivalent of 0.05% while trading in the put requires a negative reduction (increase) with −0.7% of the initial wealth.

The Sharpe-ratios based on the realized daily returns of the investment portfolios are collected in Table 3, where the results for the naive strategy are included as well.

A note on the interpretation of these results is in order here.

- First, when trading in a financial market with a call option representing the derivative, we see a positive *realized* certainty equivalent when comparing to Liu's optimal strategy for a bond-stock-only market. This holds both when trading over a long horizon of 15 years and shorter periods of two years, whereas the short-term trading shows much smaller welfare gains as predicted by the expected certainty equivalents.
- On the other hand, when a straddle or a put represents the underlying derivative, we observe close to zero welfare gains — trading in the put even decreases the realized performance measures beyond those realized by Liu's strategy.
- This discrepancy has nothing to do with the call-centric estimation procedure: if we re-run the experiment with lambda estimated from market put prices, $(\bar{\lambda}_1, \bar{\lambda}_2) = (1.14, -1.81)$, we still see the highest realized wealth for the call option, $\mathscr{W}_{T=15y} = 2{,}242$, while the put option remains suboptimal to Liu's strategy: $\mathscr{W}_{T=15y} = 1{,}469$ versus $\mathscr{W}_{T=15y} = 1{,}506$. Rather, it seems that a short put position greatly amplifies losses during financial drawdowns (S&P returns exhibit a comparatively fatter left tail).

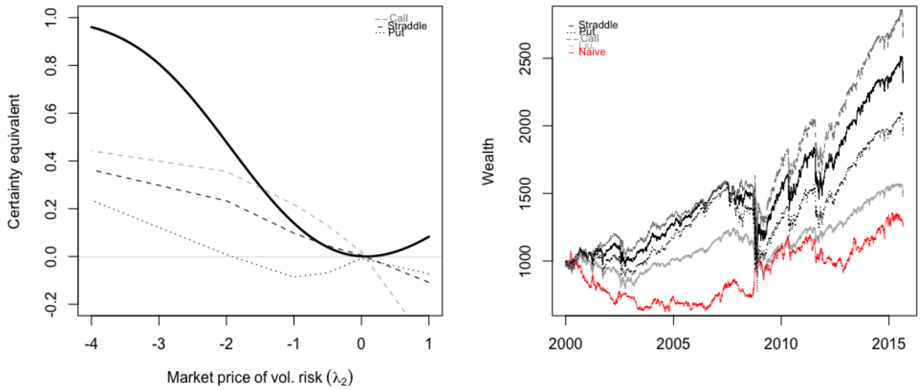

Fig. 8. Left: relative certainty equivalent in wealth for a varying market price of volatility risk $\bar{\lambda}_2$ when trading over a $T = 15$ years horizon. The black solid line shows the expected certainty equivalent as calculated from Eq. (54). The gray dashed line shows wealth-equivalents realized from trading in the call, black dashed line from trading in the straddle, while the black dotted line shows the result from the put option. The market price of risk associated with the stock index is held constant at $\bar{\lambda}_1 = 1$. Right: realized wealth paths from the setting $(\bar{\lambda}_1, \bar{\lambda}_2) = (1, -4)$, which requires the largest reduction in initial wealth for the option strategies to be equivalent Liu's strategy (both in predicted and realized equivalents, as according to the left figure).

These observations go in line with the fact that trading strategies for long horizons — and thereby expected certainty equivalents — are sensitive to the market price of volatility risk parameter, as pointed out by Larsen and Munk (2012), and Liu and Pan (2003). Even more so: if we repeat the $T = 15$ experiment for different values of the market price of volatility risk, $\bar{\lambda}_2 \in [-4, 1]$, whilst keeping a fixed $\bar{\lambda}_1 = 1$, we obtain what is shown in the left pane of Fig. 8. The results are congruent: as predicted by the expected wealth-equivalent, we see an increased level of relative gain from permitted trading in the derivatives with the highest win from the call option. For negative values of the market price of volatility risk, both the call and the straddle exhibit an increased level with the magnitude of $\bar{\lambda}_2$, whilst the put requires $\bar{\lambda}_2 < -2$ to realize a positive wealth-equivalent. Note, however, that we are far from reaching the predicted levels of certainty equivalents. For a positive market price of volatility risk, we do even end up on the negative side, in contrast to the promising prediction by the expected certainty equivalent.

- Secondly, if we take a closer look at the setting which yields the largest wealth-equivalent, $(\bar{\lambda}_1, \bar{\lambda}_2) = (1, -4)$, again with an initial wealth $w_0 = 1{,}000$ for all strategies, we obtain the realized wealth plotted in the right pane of Fig. 8. Here we observe that except for the first months of the trading period, and around the financial turmoil of 2008, the wealth generated by the derivative strategies are

Table 4. The Sharpe-ratio, mean, and standard deviation of daily portfolio returns from the strategies when trading throughout the period 2000-01-03 to 2015-08-31 under the market price of risk setting $(\bar{\lambda}_1, \bar{\lambda}_2) = (1, -4)$.

Strategy	Mean return	Std. Dev.	Sharpe	Sharpe annual
Optimal straddle	0.022%	1.29%	0.0097	0.19
Optimal call	0.025%	1.34%	0.012	0.23
Optimal put	0.017%	1.34%	0.0059	0.11
Liu's strategy	0.011%	0.64%	0.0017	0.033

generally above that of Liu's strategy. Granted: almost regardless of when we would decide to terminate our portfolios, we would come out better at the end if we engaged in the risky business of derivative trading. Realized Sharpe ratios are recorded in Table 4.

As a final note, we have completely disregarded market frictions such as bid-ask spreads and transaction fees in our analysis. Concentrating on the former, we have that the spread adds an average 1.35% to mid-market price for the straddle.[18] For the experiment with $(\bar{\lambda}_1, \bar{\lambda}_2) = (1, -4)$, the left pane of Fig. 9

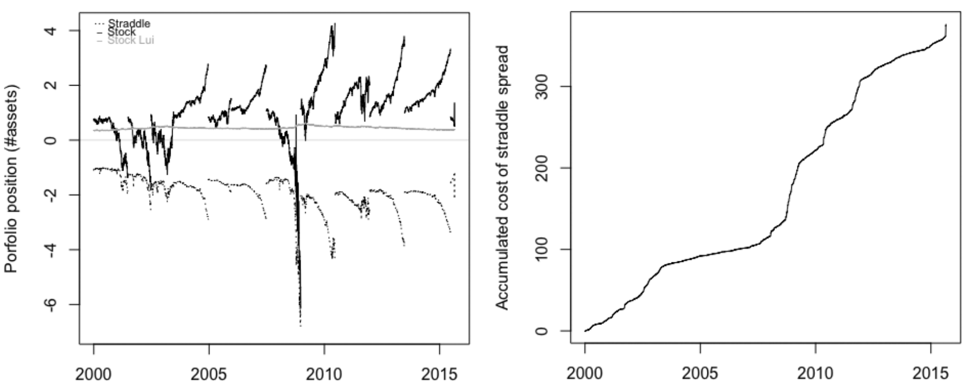

Fig. 9. Left: number of assets held in the portfolios when trading according to the optimal strategy in the straddle market and Liu's strategy. Dotted line shows the number of units of the straddle, and the black solid line shows the number of units of the stock (S&P500 index) held in the optimal portfolio. The gray solid line shows the number of stocks held in the portfolio of Liu's strategy. (Number of units held in the risk-free bank account are not included in the figure.) Right: accumulated cost charged from the bid-ask spread of the straddle price due to the trading activity of the optimal portfolio strategy.

[18]With D_t^{offer} and D_t^{bid} denoting the best offer and best bid observed on the market at time t, we calculate the spread as $\Delta_t^{\text{spread}} = D_t^{\text{offer}} - D_t^{\text{bid}}$ and the mid-market price as $D_t^{\text{mid}} = \frac{1}{2}(D_t^{\text{offer}} + D_t^{\text{bid}})$. The average relative additional cost due to the spread is hence calculated as $\frac{1}{n}\sum_t \frac{1}{2}\Delta_t^{\text{spread}}/D_t^{\text{mid}}$.

shows portfolio positions for the optimal trading in the straddle along with Liu's strategy — here plotted in the number of assets held in the portfolio. Obviously, there is a lot more trading activity when including the straddle in the portfolio — not the least in the S&P500 index itself— which would incur higher transaction costs for the optimal straddle strategy and thereby a realized wealth path inferior to that shown in Fig. 8. Indeed, if we consider the spread of the straddle price (ignoring the index-spread since it is of several magnitudes smaller) we have that the *change in units* of the optimal straddle position would accumulate to the total cost of 376.66 over the trading period, as plotted in the right pane of Fig. 9.[19] In effect, the terminal-wealth of the straddle strategy comes out at 1,727.84 when charging for the spread which corresponds to a realized certainty equivalent of 12.22%; this to be compared with the terminal wealth 2,383.84 and equivalent to 36.28% as previously recorded when transaction costs are ignored (cf. Fig. 8).

6. Conclusion

The focus of this paper has been the problem of the power-utility maximization of wealth for a risk-averse agent who consumes and invests in a financial bond-stock-option market assumed to follow a generic stochastic volatility model. In contrast to the more commonly seen solutions based on the dynamical programming principle (Larsen and Munk, 2012; Liu, 2007; and Liu and Pan, 2003), the Martingale approach was employed. Particular attention was paid to the cases where (i) the agent solely seeks to optimize her bequest from investing in the financial market, and (ii) the agent solely seeks to optimize her total consumption. Expressions for the optimal wealth process and investment/consumption policies where derived subject to the solution of a linear parabolic PDE with coefficients from the specific volatility dynamics underlying the market. In particular, the Heston model was considered as the working example with explicit formulae provided for the optimal portfolio strategy when optimizing the bequest from trading in a bond-stock-option economy.

With the optimal strategy for a restricted bond-stock economy (Liu's portfolio, Liu (2007)) as the reference point, an empirical study was then performed with the intention of measuring to which extent the inclusion of an option in the optimal portfolio would contribute to the bequest of the investor. As for the theoretical predictions based on model parameters estimated with 15 years of S&P500 data, the expectations were not particularly promising: a puny reduction with 0.18% of

[19]The portfolio position in number of units held of an asset X is $\varphi^*_{Xt} = \pi^*_{Xt}\mathscr{W}_t/X_t$ such that the cost charged from the spread of buying/selling $\Delta\varphi_{Dt}$ units of the straddle amounts to $\frac{1}{2}|\Delta\varphi_{Dt}|\Delta_t^{\text{spread}}$.

the initial capital would be the equivalent to the worth of wealth for the investor who trades in the option market over periods of ∼24 months, and a mere 1.4% with a longer horizon of ∼15 years. Indeed, the *realized* wealth obtained from optimal trading with daily rebalancing in the S&P500 index and a call option written thereon terminated above that of Liu's optimal portfolio; in terms of realized wealth equivalents, 8.1% for the 15 years' horizon and 0.65% for the 24 months' experiment. Similarly, when a straddle was considered to be constituent of the underlying market the wealth-equivalents realized slightly worse: 2.6% for the long-term trading and 0.05% for the short-term. On the other hand: in the case of a put option, the realized wealth was seen to be subordinate to that realized by Liu's strategy.

In the final part of the empirical study it was found that — in line with results previously emphasized in the literature — the market price of volatility risk (here estimated to get a model-to-market match of option prices) largely influenced the empirical trading outcome of long-term investments: ignoring market-estimates and instead trading with user-specified market prices of risk resulted in higher welfare gains with realized equivalents of 44.24% for the call, 36.29% for the straddle, and 23.54% for the put. Even if the empirical results were far from the predicted implied wealth-equivalent of 95.67%, they nonetheless established that investors who trade in options have something to gain from this risky business — a conclusion which also extends to the case of transaction costs.

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Appendix A. Proofs

A.1. Proof of Lemma 1

Proof. A hands-on differentiation of (42) with respect to the underlying, s, and subsequent algebraic simplification is a tedious exercise better avoided. A subtler, but arguably simpler argument may be presented by noting that the valuation formula is first-order homogenous in the variable pair (s, K), i.e.,

$$\begin{aligned}&\text{HestonVanilla}(\kappa, \theta, \sigma_v, \rho, \bar{\lambda}_1, \bar{\lambda}_2, r, v, as, aK, \tau', \phi) \\ &\quad = a \cdot \text{HestonVanilla}(\kappa, \theta, \sigma_v, \rho, \bar{\lambda}_1, \bar{\lambda}_2, r, v, s, K, \tau', \phi),\end{aligned}$$

for any $a \in \mathbb{R}$. From this fact we may invoke *Euler's Theorem of Homogenous Functions*[20] to deduce that D can be written on the form

$$D = s\partial_s D + K\partial_K D. \tag{A.1}$$

Comparing (42) with (A.1) it is tempting to deduce that $\partial_s D = \phi Q_1(\phi)$, yet some care must be taken here. Specifically, with two terms present in the equation we can add any arbitrary component to one term, as long as we cancel it through a corresponding subtraction to the other term, thus rendering a direct comparison meaningless. To overcome this hurdle, we therefore demonstrate that $\partial_K D = -e^{-r\tau'}\phi Q_2(\phi)$, whence the option delta follows by uniqueness. This, however, is a straightforward exercise in Breeden and Litzenberger (1978),

$$\begin{aligned}\partial_K D &= e^{-r\tau'}\partial_K \mathbb{E}^{\mathbb{Q}}_{t,s,v}[[\phi(S_{T'} - K)]^+] \\ &= e^{-r\tau'}\phi\partial_K \mathbb{E}^{\mathbb{Q}}_{t,s,v}[(S_{T'} - K)\mathbf{1}\{\phi S_{T'} \geq \phi K\}] \\ &= -e^{-r\tau'}\phi\mathbb{E}^{\mathbb{Q}}_{t,s,v}[\mathbf{1}\{\phi S_{T'} \geq \phi K\}] \\ &= -e^{-r\tau'}\phi\mathbb{Q}_{t,s,v}(\phi S_{T'} \geq \phi K) \\ &= -e^{-r\tau'}\phi Q_2(\phi),\end{aligned}$$

where the last step in the derivation follows immediately from the generalized option pricing formula (Björk, 2009, Proposition 26.11). Specifically, the functional form of Heston's pricing formula, (43), has been chosen to reflect the general pricing equation

$$D_t = \phi\{s\mathbb{S}(\phi S_{T'} \geq \phi K|\mathcal{F}_t) - e^{-r\tau'}K\mathbb{Q}(\phi S_{T'} \geq \phi K|\mathcal{F}_t)\}, \tag{A.2}$$

where $\mathbb{S}$ is the stock measure defined through $d\mathbb{S} = e^{-ru}(S_u/S_0)d\mathbb{Q}$.

Equation (46) follows immediately from differentiating (42) with respect to v. Note that the result is independent of ϕ. □

[20]Recall that the function $g : \mathbb{R}^2 \mapsto \mathbb{R}$ is said to be homogenous of degree n if $g(ax_1, ax_2) = a^n g(x_1, x_2)$. Let $x_1' = nx_1$ and $x_2' = nx_2$, then we find upon differentiating g with respect to a that $na^{n-1}g = \partial_{x_1'} g \partial_a x_1' + \partial_{x_2'} g \partial_a x_2' = x_1 \partial_{ax_1} g + x_2 \partial_{ax_2} g$. In particular, upon setting $a = 1$ we get Euler's result for homogenous functions: $ng = x_1 \partial_{x_1} g + x_2 \partial_{x_2} g$.

A.2. Proof of Lemma 2

Proof. Substituting in the relevant parametric specifications (38), (41), into the governing PDE (33) we find

$$\begin{aligned} 0 = & -\partial_\tau H^s + [\kappa\theta - \{\kappa + (1 - 1/\gamma)\bar{\lambda}_v\}v]\partial_v H^s + \frac{1}{2}\sigma_v^2 v \partial_{vv}^2 H^s \\ & + \frac{1-\gamma}{2\gamma^2}(\bar{\lambda}_1^2 + \bar{\lambda}_2^2)vH^s, \end{aligned} \tag{A.3}$$

subject to the initial condition $H^s(0, v) = 1$, where we have invoked the time transformation $t \mapsto \tau$. Since the coefficients are linear functions of v, we form the ansatz that the solution is of an exponential affine form. Thus, upon substituting (48) into (A.3) and using the fact that the expression should hold for any value of v we find the coupled ODEs,

$$\dot{B}^s(\tau) = \frac{1}{2}\sigma_v^2 B^s(\tau)^2 - \{\kappa + (1 - 1/\gamma)\bar{\lambda}_v\}B^s(\tau) + \frac{1-\gamma}{2\gamma^2}(\bar{\lambda}_1^2 + \bar{\lambda}_2^2), \tag{A.4a}$$

$$\dot{A}^s(\tau) = \kappa\theta B^s(\tau), \tag{A.4b}$$

subject to the boundary conditions $A^s(0) = B^s(0) = 0$, where $\dot{}$ denotes the derivative with respect to τ. The first equation is Riccatian, which readily allows us to extract the solution (49).[21] Note that ω is a real number insofar as $\gamma > 1$, which we assume to be the case. As for the function A^s, we observe that (A.4b) can we written as $A^s(\tau) = \kappa\theta \int_0^\tau B^s(t)dt$. Performing this tedious integration we get the desired result. □

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[21]Recall that the generic Riccati equation $dy/dx(x) = ay(x)^2 + by(x) + c$ with $y(0) = 0$ has the solution $y(x) = [2c(e^{\delta x} - 1)]/[(\delta - b)(e^{\delta x} - 1) + 2\delta]$, where $\delta \equiv \sqrt{b^2 - 4ac}$ assuming $b^2 > 4ac$.

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