A trajectorial interpretation of entropy dissipation and a non intrinsic Bakry Emery criterion for diffusion processes

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June 2012

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Outline

1. Motivation and framework: long-time convergence of Markov processes using functional inequalities (Bakry Emery criterion)
2. A pathwise (probabilistic) point of view of entropy dissipation
3. Pathwise entropy dissipation for diffusion processes
4. Convergence to equilibrium: a new (non intrinsic) Bakry-Emery criterion for exponential trend to equilibrium
5. Conclusions, current and future work

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Diffusion process and entropy

$b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \otimes d'}$, W $d'-$dimensional B.M.

$$dX_t = b(X_t) + \sigma(X_t) dW_t \quad \in \mathbb{R}^d$$

Markov diffusion with invariant density p_∞.

- $U : [0, \infty) \to \mathbb{R}$ convex and such that *inf* $U > -\infty$.
- $H_U(p|q) = \begin{cases} \int_E U\left(\frac{dp}{dq}(x)\right) dq(x) & \text{if } p \ll q \\ +\infty \text{ otherwise.} \end{cases}$ $U-$ relative entropy.

Examples: $U(r) = r \ln(r)$, $U(r) = (r-1)^2$, $U(r) = |r-1|$.

Entropy decrease: an analytic point of view

Classic argument :

$$\frac{d}{dt}H_U(p_t|p_\infty) = \dots \text{ integ. by parts, PDE} \dots = -\frac{1}{2}I_U(p_t|p_\infty)$$

where $I_U(p_t|p_\infty) =$

$$\int_{\mathbb{R}^d} U''\left(\frac{p_t}{p_\infty}(x)\right)\left(\nabla^*\left[\frac{p_t}{p_\infty}\right]a(t,\cdot)\nabla\left[\frac{p_t}{p_\infty}\right]\right)(x)p_\infty(x)dx \geq 0$$

is U-Fisher information ($a = \sigma\sigma^*$) or dissipation of U-entropy.

Bakry-Emery approach to long-time convergence rate

- Bakry Emery Curvature Dimension Criterion (BEC) involving $\mathcal{L}, \Gamma, \Gamma_2$ provide conditions for

$$\frac{d}{dt} I_U(p_t|p_\infty) \leq -2\lambda I_U(p_t|p_\infty)$$

to hold for some $\lambda > 0$. Then $I_U(p_s|p_\infty) \leq e^{-2\lambda s} I_U(p_0|p_\infty)$.

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$$H_U(p_s|p_\infty) - \lim_{t\to\infty} H_U(p_t|p_\infty) = \int_s^\infty I_U(p_r|p_\infty)dr \leq \frac{e^{-2\lambda s}}{2\lambda} I_U(p_0|p_\infty).$$

- if $\lim_{t\to\infty} = 0$, $s = 0 \Rightarrow$ Convex-Sobolev ineq. :

$$H_U(p_0|p_\infty) \leq \frac{1}{2\lambda} I_U(p_0|p_\infty) \quad \Rightarrow H_U(p_t|p_\infty) \leq e^{-\lambda t} H_U(p_0|p_\infty).$$

- $U(r) = r \ln r \Rightarrow$ Log-Sobolev, $U(r) = (r-1)^2 \Rightarrow$ Poincaré. Both imply $\|p_t - p_\infty\|_{TV} \to 0$ exponentially fast.

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A pathwise probabilistic viewpoint to entropy dissipation

Notation:

- $(X_t : t \geq 0)$ continuous-time Markov process with values in some space E.
- $(X_t^{P_0}, t \geq 0)$ and $(X_t^{Q_0}, t \geq 0)$ version of (X_t) with $X_0^{P_0} \sim P_0$ and $X_0^{Q_0} \sim Q_0$ respectively.
- $P_t := \mathcal{L}(X_t^{P_0})$ and $Q_t := \mathcal{L}(X_t^{Q_0})$.

Proposition 1 .

If for some $t \geq 0$, $P_t \ll Q_t$, then :

- $\mathcal{L}(X_r^{P_0} : r \geq t) \ll \mathcal{L}(X_r^{Q_0} : r \geq t)$ with density $\frac{dP_t}{dQ_t}(X_t^{Q_0})$
- for all $s \geq t$, $P_s \ll Q_s$,
- $\left(\frac{dP_s}{dQ_s}(X_s^{Q_0})\right)_{s \geq t}$ is a backward martingale with respect to the filtration $\mathcal{F}_s = \sigma(X_r^{Q_0}, r \geq s)$.

$\Rightarrow \lim_{s\to\infty} \frac{dP_s}{dQ_s}(X_s^{Q_0}) = \mathbb{E}(\frac{dP_t}{dQ_t}(X_t^{Q_0}) \big| \cap_{s\geq 0} \mathcal{F}_s)$ exists *a.s.* and in L^1.

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Entropy decrease:

Corollary

If $H_U(P_t|Q_t) < +\infty$ for some $t \geq 0$, then

- $\left(U(\frac{dP_s}{dQ_s}(X_s^{Q_0}))\right)_{s\geq t}$ is a u.i. backward $\mathcal{F}_s-$ submartingale

$$\Rightarrow \quad s \in [t,\infty) \mapsto H_U(P_s|Q_s) = \mathbb{E}\left(U\left(\frac{dP_s}{dQ_s}(X_s^{Q_0})\right)\right) \in [0,\infty)$$

  non-increasing.

- $$\lim_{s\to\infty} H_U(P_s|Q_s) = \mathbb{E}\left(U\left(\lim_{s\to\infty}\frac{dP_s}{dQ_s}(X_s^{Q_0})\right)\right) < \infty.$$

  In particular $\lim_{s\to\infty} H_U(P_s|Q_s) = U(1)$ if tail σ-field $\cap_{s\geq 0}\mathcal{F}_s$ is trivial a.s.

E.g.: trivial tail σ-field if $(X_t)_{t\geq 0}$ Feller, positive recurrent with bi-continuous transition densities and $p_\infty > 0$.

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E.g.: trivial tail σ-field if $(X_t)_{t\geq 0}$ Feller, positive recurrent with bi-continuous transition densities and $p_\infty > 0$.

Proof of Prop. 1: if $P_t \ll Q_t$,

$$\mathbb{E}(f(X_r^{P_0}, r \geq t)) = \int_E \mathbb{E}^{t,x}(f(X_r, r \geq t))P_t(dx) =$$

$$\int_E \mathbb{E}^{t,x}\left(f(X_r, r \geq t)\frac{dP_t}{dQ_t}(X_t)\right) Q_t(dx) = \mathbb{E}(f(X_r^{Q_0}, r \geq t)\frac{dP_t}{dQ_t}(X_t^{Q_0})).$$

For $s \geq t$ taking $f(X_r^{P_0}, r \geq t) = f(X_s^{P_0}) \quad \Rightarrow P_s \ll Q_s$.

Moreover

$$\mathbb{E}\left(f(X_r^{P_0}, r \geq s)\right) = \mathbb{E}\left(f(X_r^{Q_0}, r \geq s)\frac{dP_t}{dQ_t}(X_t^{Q_0})\right)$$

and also

$$\mathbb{E}\left(f(X_r^{P_0}, r \geq s)\right) = \mathbb{E}\left(f(X_r^{Q_0}, r \geq s)\frac{dP_s}{dQ_s}(X_s^{Q_0})\right).$$

Diffusion process:

$$dX_t = b(t, X_t) + \sigma(t, X_t) dW_t \quad \in \mathbb{R}^d$$

$b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$,
$\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \otimes d'}$, $a = \sigma\sigma^* : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \otimes d}$.
W $d'-$ dimensional BM.

GOAL: describe $\left(U(\frac{dP_s}{dQ_s}(X_s^{Q_0})) \right)_{0 \leq s \leq T}$ for fixed $T > 0$.

We revert time to work with forward martingales:

- $(Y_t : t \leq T) := (X_{T-t}^{Q_0}, t \leq T)$,
  $\mathcal{G}_t := \sigma(Y_s, 0 \leq s \leq t)$, $t \in [0, T]$ its filtration.
- $\mathbb{Q}^{T \to 0} := \mathcal{L}(Y_t : t \leq T) = \mathcal{L}(X_{T-t}^{Q_0}, t \leq T)$
- $\mathbb{P}^{T \to 0} := \mathcal{L}(X_{T-t}^{P_0}, t \leq T)$

Then

- $P_0 \ll Q_0 \Longrightarrow \mathbb{P}^{T\to 0} \ll \mathbb{Q}^{T\to 0}$ with $\frac{d\mathbb{P}^{T\to 0}}{d\mathbb{Q}^{T\to 0}} = \frac{dP_0}{dQ_0}(Y_T)$. Define

$$D_t^T \stackrel{\text{def}}{=} \frac{d\mathbb{P}^{T\to 0}}{d\mathbb{Q}^{T\to 0}}|_{\mathcal{G}_t} = \frac{dP_{T-t}}{dQ_{T-t}}(Y_t), \quad 0 \le t \le T,$$

  $\mathbb{Q}^{T\to 0} - \mathcal{G}_t$ martingale (Girsanov density).

- **Pathwise entropy:**

$$\mathbb{H}_U(\mathbb{P}^{0\to T}|\mathbb{Q}^{0\to T}) = H_U(P_0|Q_0) = \mathbb{H}_U(\mathbb{P}^{T\to 0}|\mathbb{Q}^{T\to 0}) .$$

  Remark: Föllmer ('84), used entropy to study time-reversal of diffusions.
  Here we will use time reversal of diffusion processes to study entropy.

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- $P_0 \ll Q_0 \Longrightarrow \mathbb{P}^{T\to 0} \ll \mathbb{Q}^{T\to 0}$ with $\frac{d\mathbb{P}^{T\to 0}}{d\mathbb{Q}^{T\to 0}} = \frac{dP_0}{dQ_0}(Y_T)$. Define

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  Remark: Föllmer ('84), used entropy to study time-reversal of diffusions.
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Conditions for time reversal of diffusions [Föllmer 84, Haussmann, Pardoux 86, Millet, Nualart, Sanz 89]

- σ, b locally Lipschitz + exp. integ. of derivatives
- $Q_t = law(X_t^{Q_0}) = q_t(x)dx$
- $\sum_j \partial_j(a_{ij}(t,x)q_t(x))$ in $L^1_{loc}(dx\ dt)$

then $\mathbb{Q}^{T\to 0}$ solves the martingale problem

$$M_t^f := f(Y_t) - f(Y_0) - \int_0^t \frac{1}{2}\bar{a}_{ij}(s, Y_s)\partial_{ij}f(Y_s) + \bar{b}^i_{Q_0}(s, Y_s)\partial_i f(Y_s)ds$$

- $\bar{a}_{ij}(t,x) := a_{ij}(T-t,x), i,j = 1,\dots,d,$
- $\bar{b}^i_{Q_0}(t,x) = -b^i(T-t,x) + \frac{\partial_j(a_{ij}(T-t,x)q_{T-t}(x))}{q_{T-t}(x)}$ $\quad (\frac{*}{0} = 0.)$

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$\Longrightarrow$ Girsanov theory provides D_t^T.

Lemma Assume time reversibility of the diffusion property, and $P_0 \ll Q_0$. Then

- $\nabla \ln \frac{p_t}{q_t}$ and $\nabla \frac{p_t}{q_t}$ do have a meaning.
- Under mild assumptions setting

  $M^i_t := Y^i_t - Y^i_0 - \int_0^t \bar{b}^i_{Q_0}(s, Y_s)ds$ and

  $R := \inf\{s \in [0, T] : D^T_s = 0\}$ $(\mathcal{G}_t)$-stopping time, we have

$$D^T_t = \frac{p_T}{q_T}(Y_0) + \int_0^t \nabla \left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s) \mathbf{1}_{s<R} \cdot dM_s$$

$$= \frac{p_T}{q_T}(Y_0) + \int_0^t \nabla \left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s) \mathbf{1}_{\{\frac{p_{T-s}}{q_{T-s}}(Y_s)>0\}} \cdot dM_s \quad \mathbb{Q}^{T\to 0} a.s,$$

and

$$\langle D^T \rangle_t = \int_0^t \left(\nabla \left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\right)^* \bar{a}(s, Y_s) \nabla \left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s) \mathbf{1}_{s<R}\, ds (< \infty).$$

Lemma Assume time reversibility of the diffusion property, and $P_0 \ll Q_0$. Then

- $\nabla \ln \frac{p_t}{q_t}$ and $\nabla \frac{p_t}{q_t}$ do have a meaning.
- Under mild assumptions setting $M_t^i := Y_t^i - Y_0^i - \int_0^t \bar{b}^i_{Q_0}(s, Y_s)ds$ and $R := \inf\{s \in [0,T] : D_s^T = 0\}$ $(\mathcal{G}_t)$-stopping time, we have

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D_t^T &= \frac{p_T}{q_T}(Y_0) + \int_0^t \nabla \left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{s<R} \cdot dM_s \\
&= \frac{p_T}{q_T}(Y_0) + \int_0^t \nabla \left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{\{\frac{p_{T-s}}{q_{T-s}}(Y_s)>0\}} \cdot dM_s \quad \mathbb{Q}^{T\to 0} a.s,
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$$

- $\mathbb{Q}^{T\to 0}-$a.s. $\forall t \in [0,T]$,

$$D_t^T = \mathbf{1}_{\{t<\tau\}} \frac{dp_T}{dq_T}(Y_0) \times$$
$$\exp\left\{ \int_0^t \nabla \left[\ln \frac{p_{T-s}}{q_{T-s}} \right](Y_s) \cdot dM_s \right.$$
$$\left. - \frac{1}{2} \int_0^t \left(\nabla \left[\ln \frac{p_{T-s}}{q_{T-s}} \right](Y_s) \right)^* \bar{a}(s, Y_s) \nabla \left[\ln \frac{p_{T-s}}{q_{T-s}} \right](Y_s) ds \right\}.$$

where

$$\tau := \inf \left\{ t \in [0,T] : \right.$$
$$\left. \int_0^t \left(\nabla \left[\ln \frac{p_{T-s}}{q_{T-s}} \right](Y_s) \right)^* \bar{a}(s, Y_s) \nabla \left[\ln \frac{p_{T-s}}{q_{T-s}} \right](Y_s) ds = \infty \right\},$$

and one has $R = \tau \wedge \tau^o$ where $\tau^o := 0 \cdot \mathbf{1}_{D_0^T = 0} + \infty \cdot \mathbf{1}_{D_0^T > 0}$

(related ideas: stochastic construction of Nelson processes Cattiaux-Léonard,....$\sim$ 95)

Stochastic $U-$entropy dissipation formula

Theorem 1 (F.-Jourdain)

Assume $H_U(P_0|Q_0) < \infty$ (+) mild assumptions. The $\mathbb{Q}^{T\to 0} - \mathcal{G}_t$ submartingale $(U(D_t^T))_{t\in[0,T]}$ has Doob-Meyer decomposition

$$U(D_t^T) = U(D_0^T) + \int_0^t U'_-(D_s^T)\nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{s<R}\cdot dM_s + \frac{1}{2}\int_{(0,+\infty)} L_t^r(D^T)U''(dr) - 1_{\{0<R\leq t\}}\Delta U(0),$$

where $L_t^r(D^T)$ is the local time at level $r \geq 0$ and time t of $(D_s^T)_{s\in[0,T]}$ and $\Delta U(0) = \text{lím}_{x\to 0^+} U(x) - U(0) \leq 0$.

In particular, if U is continuous on $[0,+\infty)$ and C^2 on $(0,+\infty)$, $\forall t \in [0,T]$

$$U(D_t^T) = U(D_0^T) + \int_0^t U'(D_s^T)\nabla\left[\frac{p_{T-s}}{q_{T-s}}\right](Y_s)\mathbf{1}_{s<R}\cdot dM_s$$
$$+\frac{1}{2}\int_0^t U''\left(\frac{p_{T-s}}{q_{T-s}}(Y_s)\right)\left(\nabla^*\left[\frac{p_{T-s}}{q_{T-s}}\right]\bar{a}(s,\cdot)\nabla\left[\frac{p_{T-s}}{q_{T-s}}\right]\right)(Y_s)\mathbf{1}_{s<R}ds.$$

Taking expectation w.r.t. $\mathbb{Q}^{T\to 0}$, $t = 0$, we get

U- entropy dissipation:

$$H_U(P_T|Q_T) = H_U(P_0|Q_0) - \frac{1}{2}\int_{(0,+\infty)} \mathbb{E}^{T\to 0}\left(L_T^r(D^T)\right) U''(dr) + \Delta U(0)\mathbb{Q}^{T\to 0}(0 < R \leq T).$$

Moreover

$$H_U(P_T|Q_T) = H_U(P_0|Q_0) - \frac{1}{2}\int_{(0,+\infty)} \mathbb{E}^{0\to T}\left(L_T^r\left(\frac{p_\cdot}{q_\cdot}(X^Q)\right)\right) U''(dr) + \Delta U(0)\mathbb{Q}^{0\to T}(0 < S \leq T)$$

if $\frac{p_s}{q_s}(X_s^Q)$ semimartingale, where $S := \sup\{t \geq 0 : \frac{p_t}{q_t}(X_t^Q) = 0\} = \inf\{t \geq 0 : \frac{p_t}{q_t}(X_t^Q) > 0\}$ is stopping time.

U- entropy dissipation:

In particular if U is continuous on $[0,+\infty)$ and C^2 on $(0,+\infty)$, we recover the well known formula, for arbitrary initial laws:

$$H_U(P_T|Q_T) = H_U(P_0|Q_0)$$
$$-\frac{1}{2}\int_0^T \underbrace{\int_{\{\frac{p_s}{q_s}(x)>0\}} U''\left(\frac{p_s}{q_s}(x)\right)\left(\nabla^*\left[\frac{p_s}{q_s}\right]a(s,\cdot)\nabla\left[\frac{p_s}{q_s}\right]\right)(x)q_s(x)dx\,ds}_{I_U(p_s|q_s)\text{ U-Fisher information}}$$

Corollary: Dissipation of total variation

1) For the choice $U(x) = |x - 1|$,

$$\|P_T - Q_T\|_{\mathrm{TV}} = \|P_0 - Q_0\|_{\mathrm{TV}} - \mathbb{E}^{0\to T}\left(L_T^1\left(\frac{p_\cdot}{q_\cdot}(X^Q)\right)\right).$$

2) (+) mild additional assumptions : $\forall t \geq 0$,

$$\|P_T - Q_T\|_{\mathrm{TV}} = \|P_0 - Q_0\|_{\mathrm{TV}} + \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} \widetilde{sign}\left(\frac{p_s}{q_s} - 1\right)(x)\nabla \cdot \left[a(s,x)\nabla\left[\frac{p_s}{q_s}\right](x)q_s(x)\right] dxds$$

where $\widetilde{sign}(r) = -\mathbf{1}_{(-\infty,0)}(r) + \mathbf{1}_{(0,\infty)}(r)$ and the integral is non positive.

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When $q_0 = p_\infty$ we have :

$$\frac{d}{dt} H_U(p_t|p_\infty) = -\frac{1}{2} I_U(p_t|p_\infty)$$

where U-Fisher information is $I_U(p_t|p_\infty) =$

$$\int_{\{\frac{p_t}{p_\infty}(x)>0\}} U''\left(\frac{p_t}{p_\infty}(x)\right)\left(\nabla^*\left[\frac{p_t}{p_\infty}\right] a(t,\cdot)\nabla\left[\frac{p_t}{p_\infty}\right]\right)(x)p_\infty(x)dx.$$

As in Bakry Emery criterion, we look for

$$\frac{d}{dt} I_U(p_t|p_\infty) \leq -2\lambda I_U(p_t|p_\infty)$$

to hold for some $\lambda > 0$

A Bakry Emery type criterion relying on the choice of σ

We assume

- regular time-homogeneous coefficients $\sigma(x)$ and $b(x)$ $(+)$ growth conditions .
- $\frac{Q_0(dx)}{dx} = p_\infty(x) > 0$ regular invariant (non necessarily reversible) density.
- $\mathbb{Q}^{T\to 0} = \mathbb{P}_\infty^{T\to 0}$ stationary time reverted law.
- $U : [0,\infty) \to \mathbb{R}$ convex of class C^4 on $(0,+\infty)$, continuous on $[0,+\infty)$, such that $U(1) = U'(1) = 0$ and

$$(U^{(3)}(r))^2 \leq \frac{1}{2} U''(r) U^{(4)}(r)$$

($\Leftrightarrow 1/(U'')$ concave. "Admissible entropies", Arnold et al. 01, Chafaï 04)

Theorem 2 (F.-Jourdain 2011)

Let

$$\Theta_{ll'} = \sigma_{l'i}[\bar{b}_m \partial_m \sigma_{li} + \frac{1}{2} a_{mk} \partial_{mk} \sigma_{li}] - a_{kl'} \partial_k \bar{b}_l$$
$$+ (\sigma_{ki} a_{ml'} - \sigma_{l'i} a_{mk}) \partial_m \sigma_{li} \partial_k \ln(p_\infty) + \partial_k[(\sigma_{ki} a_{ml'} - \sigma_{l'i} a_{mk}) \partial_m \sigma_{li}]$$

If Θ satisfies the "non-intrinsic Bakry Emery Criterion "

$$NIBEC) \quad \exists \lambda > 0, \ \forall x \in \mathbb{R}^d, \ \frac{1}{2}(\Theta + \Theta^*)(x) \geq \lambda a(x)$$

then $\frac{d}{dt} I_U(p_t|p_\infty) \leq -2\lambda I_U(p_t|p_\infty)$ and $H_U(p_t|p_\infty) \to$ at exponential rate λ as $t \to \infty$.

If moreover a is strictly elliptic OR hypoellipticity conditions hold, then the limit is 0 and the convex Sobolev inequality $H_U(p|p_\infty) \leq \frac{1}{2\lambda} I_U(p|p_\infty)$ holds for all probability density p on $\mathbb{R}^d$.

Explicitly:

$$\frac{1}{2}(\Theta + \Theta^*) = -\frac{1}{2} b_m \partial_m a_{ll'}$$

$$+ \frac{1}{2}(a_{kl'}\partial_k b_l + a_{kl}\partial_k b_{l'}) - \frac{1}{4} a_{mk}\partial_{mk} a_{ll'} - \frac{1}{2}(a_{kl'}\partial_{kj} a_{lj} + a_{kl}\partial_{kj} a_{l'j})$$

$$- a_{kl} a_{jl'} \partial_{kj} \ln(p_\infty) - \frac{1}{2}(a_{kl}\partial_k a_{l'j} + a_{kl'}\partial_k a_{lj})\partial_j \ln(p_\infty)$$

$$- \frac{1}{2} a_{mk} \partial_m \sigma_{li} \partial_k \sigma_{l'i}$$

$$+ \frac{1}{2}\sigma_{ki}(\partial_m \sigma_{li} a_{ml'} + \partial_m \sigma_{l'i} a_{ml})\partial_k \ln(p_\infty) + \frac{1}{2}\partial_k[\sigma_{ki}(\partial_m \sigma_{li} a_{ml'} + \partial_m \sigma_{l'i} a_{ml})]$$

Remark

i) Θ cannot be written without using square root σ and depends on its choice (compare to BEC only depending on $\mathcal{L}$).

ii) *a* non singular is not needed.

iii) In case $a = 2\nu I_d$ and $b = -(\nabla V + F)$ with F such that $\nabla.(e^{-V/\nu}F) = 0$, then $p_\infty \propto e^{-V/\nu}$, $\bar{b} = -b + 2\nu\nabla \ln p_\infty = -\nabla V + F$ and $\Theta = 2\nu(\nabla^2 V - \nabla F)$.

For the choice $\sigma = \sqrt{2\nu}I_d$, then *NIBEC)* writes

$$\exists\lambda > 0,\ \forall x \in \mathbb{R}^d,\ \nabla^2 V(x) - \frac{\nabla F + \nabla F^*}{2}(x) \geq \lambda I_d$$

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Example

$d = 2$ and for each $(x_1, x_2) \in \mathbb{R}^2$,

$$a(x_1, x_2) = I_2, \quad \text{and} \quad b(x_1, x_2) = -\nabla V(x_1, x_2)$$

with V convex C^2 potential

$$V(x_1, x_2) := |x_1|^2 + |x_1 - x_2|^{2+\alpha} + |x_2|^{2+\alpha}$$

for some $\alpha \in (0, 1)$. Distribution : $p_\infty \propto e^{-2V}$

- Classic Bakry Emery criterion $\nabla^2 V \geq \lambda I_2$ for $\lambda > 0$ fails at $(0, 0)$.
- Holley-Stroock perturbation argument provides Log-Sobolev inequality
- Arnold,Carlen & Ju (08) obtain convex Sob.inequality, first for a non-reversible diffusion with same stationary law (add drift F such that $\nabla \cdot (F p_\infty) = 0$), then come back to $F = 0$.

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With our method

choose square root σ of the identity matrix of the form

$$\sigma(x_1,x_2) = \begin{pmatrix} \cos\phi(x_1,x_2) & \sin\phi(x_1,x_2) \\ -\sin\phi(x_1,x_2) & \cos\phi(x_1,x_2) \end{pmatrix}$$

(law of the diffusion processes not modified). Take

$$\phi(x_1,x_2) = -\varepsilon\varphi_\varepsilon(x_1)\varphi_\varepsilon(x_2), \quad (x_1,x_2)\in\mathbb{R}^2$$

where $\varphi_\varepsilon(s) = \varepsilon\varphi(s/\varepsilon)$ and $\varphi(s) = \begin{cases} s & \text{if } |s|\le 1, \\ 0 & \text{if } |s|\ge 2. \end{cases}$

$$\frac{1}{2}(\Theta+\Theta^*) = \nabla^2 V - \frac{1}{2}|\nabla\phi|^2 I_2 + \begin{pmatrix} \partial_{12}\phi & \frac{\partial_{22}\phi-\partial_{11}\phi}{2} \\ \frac{\partial_{22}\phi-\partial_{11}\phi}{2} & -\partial_{12}\phi \end{pmatrix}$$
$$+ \begin{pmatrix} -2\partial_1\phi\partial_2 V & \partial_1\phi\partial_1 V - \partial_2\phi\partial_2 V \\ \partial_1\phi\partial_1 V - \partial_2\phi\partial_2 V & 2\partial_2\phi\partial_1 V \end{pmatrix}$$

$\frac{1}{2}(\Theta + \Theta^*)$

$$= \begin{cases} \nabla^2 V + \begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} + O(\varepsilon^3) \geq \begin{pmatrix} 2-\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} + O(\varepsilon^3) & \text{on } B_\varepsilon \\ \nabla^2 V + O(\varepsilon) \geq (2+\alpha)(1+\alpha)\varepsilon^\alpha I_2 + o(\varepsilon^\alpha) & \text{on } B_{2\varepsilon} \backslash B_\varepsilon \\ \geq I_2((2+\alpha)(1+\alpha)(2\varepsilon)^\alpha) \wedge 2 & \text{on } B_\varepsilon^c \end{cases}$$

$\Rightarrow$ **NIBEC)** holds for sufficiently small $\varepsilon > 0$

Proof of Th. 2 . Step 1

Proposition 2:

Let $\rho_t := p_{T-t}/p_\infty$ and all functions be computed at (t, Y_t). Then

$$d\left[U''(\rho_t)\nabla^*\rho_t a\nabla\rho_t\right](Y_t) = tr(\Lambda\Gamma)dt + U''(\rho)\bar{\theta}dt + d\hat{M}$$

with Λ and Γ square matrices $\Lambda := \begin{bmatrix} U''(\rho) & U^{(3)}(\rho) \\ U^{(3)}(\rho) & \frac{1}{2}U^{(4)}(\rho) \end{bmatrix} \geq 0$

$$\Gamma := \begin{bmatrix} \nabla^*(\sigma_{\bullet i}\cdot\nabla\rho)a\nabla(\sigma_{\bullet i}\cdot\nabla\rho) & (\sigma_{\bullet i}\cdot\nabla\rho)\nabla^*\rho\, a\nabla(\sigma_{\bullet i}\cdot\nabla\rho) \\ (\sigma_{\bullet i}\cdot\nabla\rho)\nabla^*\rho\, a\nabla(\sigma_{\bullet i}\cdot\nabla\rho) & |\nabla^*\rho a\nabla\rho|^2 \end{bmatrix},$$ and

$$\bar{\theta} = 2\Big\{\partial_{l'}\rho\partial_l\rho\left(\sigma_{l'i}\left[\bar{b}_m\partial_m\sigma_{li} + \frac{1}{2}a_{mk}\partial_{mk}\sigma_{li}\right] - a_{ml'}\partial_m\bar{b}_l\right) + \left[\sigma_{l'i}a_{mk} - \sigma_{ki}a_{ml'}\right]\partial_{l'}\rho\partial_m\sigma_{li}\partial_{kl}\rho\Big\}$$

- **Goal**: bound (in mean) $tr(\Lambda\Gamma)dt + U''(\rho)\bar{\theta}dt + d\hat{M}$ **from below** by

$$2\lambda\left[U''(\rho_t)\nabla^*\rho_t a\nabla\rho_t\right](Y_t)$$

- Remark: all terms **but** Γ_{11} equal to those in Arnold et al. (08). **But our** Γ_{11} **cannot be written without making use of** σ**.**
- **In our case** $det(\Gamma) \geq 0$ by **Cauchy-Schwarz**:

$$\begin{aligned}((\sigma_{\bullet i}\cdot\nabla\rho)\nabla^*\rho\; a\nabla(\sigma_{\bullet i}\cdot\nabla\rho))^2 &= ((\sigma_{\bullet i}\cdot\nabla\rho)\sigma^*\nabla\rho.\sigma^*\nabla(\sigma_{\bullet i}\cdot\nabla\rho))^2\\ &\leq \sum_i(\sigma_{\bullet i}\cdot\nabla\rho)^2|\sigma^*\nabla\rho|^2\sum_i|\sigma^*\nabla(\sigma_{\bullet i}\cdot\nabla\rho)|^2\\ &= |\nabla^*\rho a\nabla\rho|^2\times\nabla^*(\sigma_{\bullet i}\cdot\nabla\rho)a\nabla(\sigma_{\bullet i}\cdot\nabla\rho).\end{aligned}$$

We deduce $\Gamma \geq 0$ and since $\Lambda \geq 0$, we get $tr(\Lambda\Gamma) \geq 0$

$$\Longrightarrow d\left[U''\rho)\nabla^*\rho a\nabla\rho\right] \geq U''(\rho)\bar{\theta}dt + d\hat{M}$$

Remark: $\bar{\theta}$ **depends on** $\partial_{kl}\rho$

Sketch of proof of Prop. 2:

- Stochastic flow
  $d\xi_t^i(x) = \sigma_{ik}(\xi_t(x))d\bar{W}_t^k + \bar{b}_i(\xi_t(x))dt, \quad (t,x) \in [0,T) \times \mathbb{R}^d,$
  $\xi_0(x) = x \quad\quad (\xi_t(Y_0) = Y_t).$

- $D_t^T = \rho_t(\xi_t(Y_0))$ desintegrates into the continuous $\mathcal{G}_t - \mathbb{P}_\infty^{T\to 0}-$ martingales $(D_t(x) = \rho_t(\xi_t(x)))_{t\in[0,T]}, x \in \mathbb{R}^d$, satisfying

$$dD_t(x) := [\sigma_{ik}\partial_i\rho](t, \xi_t(x))d\bar{W}_t^k \quad , \quad D_0(x) = \frac{p_T}{p_\infty}(x) = \rho_0(x).$$

- Then $\nabla_x[D_t(x)] = \nabla_x\xi_t(x) \cdot \nabla[\rho_t](\xi_t(x))$
  Use SDE for $d(\nabla_x\xi_t(x))^{-1}$ and Itô product rule to get $\nabla[\rho_t](\xi_t(x))$.

- For $U''(\rho)\nabla^*\rho_t a\nabla\rho_t$ use Itô product rule product rule like this:

$$U''(\rho)(\sigma^*\nabla\rho_t).(\sigma^*\nabla\rho_t)$$

(otherwise, one can recover the A.C.J criterion).

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Proof of Th. 2 . Step 2

$$
\begin{aligned}
&d[U''(\rho)\nabla^*\rho a\nabla\rho](t,Y_t)\\
&\qquad \geq d\hat{M}_t + 2\partial_{kl}\rho\,\partial_{l'}\rho U''(\rho)\,[\sigma_{l'i}a_{mk}-\sigma_{ki}a_{ml'}]\,\partial_m\sigma_{li}dt\\
&\qquad + 2U''(\rho)\partial_{l'}\rho\partial_l\rho\left(\sigma_{l'i}\left[\bar{b}_m\partial_m\sigma_{li}+\frac{1}{2}a_{mk}\partial_{mk}\sigma_{li}\right]-a_{ml'}\partial_m\bar{b}_l\right)dt.
\end{aligned}
$$

Take expectations under $\mathbb{P}_\infty^{T\to 0}$. Term with 2nd order derivatives:
$\int_{\mathbb{R}^d}\partial_{kl}\rho\,\partial_{l'}\rho U''(\rho)\,[\sigma_{l'i}a_{mk}-\sigma_{ki}a_{ml'}]\,\partial_m\sigma_{li}p_\infty dx =$

$$
\begin{aligned}
&-\int_{\mathbb{R}^d}\partial_l\rho\partial_{l'}\rho U''(\rho)\,[\sigma_{l'i}a_{mk}-\sigma_{ki}a_{ml'}]\,\partial_m\sigma_{li}\partial_k(\ln p_\infty)p_\infty dx\\
&\qquad -\int_{\mathbb{R}^d}\partial_l\rho\partial_{l'}\rho U''(\rho)\partial_k\left([\sigma_{l'i}a_{mk}-\sigma_{ki}a_{ml'}]\,\partial_m\sigma_{li}\right)p_\infty dx
\end{aligned}
$$

since

$$\partial_{kl'}\rho\,[\sigma_{l'i}a_{mk}-\sigma_{ki}a_{ml'}]=0 \text{ and}$$

$$\partial_k(U''(\rho))\partial_{l'}\rho\,[\sigma_{l'i}a_{mk}-\sigma_{ki}a_{ml'}]=U^{(3)}(\rho)\partial_k\rho\partial_{l'}\rho\,[\sigma_{l'i}a_{mk}-\sigma_{ki}a_{ml'}]=0.$$

1. Motivation and framework: long-time convergence of Markov processes using functional inequalities (Bakry Emery criterion)
2. A pathwise (probabilistic) point of view of entropy dissipation
3. Pathwise entropy dissipation for diffusion processes
4. Convergence to equilibrium: a new (non intrinsic) Bakry-Emery criterion for exponential trend to equilibrium
5. **Conclusions, current and future work**

Conclusions

- pathwise point of view provides further probabilistic insight of long-time behavior
- choice of the square root of the diffusion matrix (Hörmander vector fields) might improve bounds on the convergence rate.

Work in progress and open questions

- Current work : general method for good choice of the square root?
- Predictable optimal choice of $\sigma(t, X_t, \omega)$?

  Idea: " Lift " the SDE to orthogonal group $\Rightarrow$ SDE for (X_t, σ_t)

  (as in Cruzeiro, Malliavin & Thalmaier '04: "Geometrization" of the Milstein scheme)
- Hypocoercivity?

Thank you!