On the higher-order weak approximation of SDEs

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Abstract

Higher order weak approximation algorithms
  The weak approximation problem
  Our Higher-order scheme
  Main result
  The Algorithms
  Numerical Example

Recent developments
  Algorithm for barrier options
  Semi-closed form solutions to SDEs

Appendix/Backup
Abstract

Higher order weak approximation algorithms
Recent developments
Appendix/Backup

Background:

- A new higher-order weak approximation scheme based on [Kusuoka ’01] and [Lyons and Victoir ’04]

Objective:

- Construction of Concrete higher-order weak approximation algorithms that are:
  - Versatile, (applicable to a broad class of SDEs)
  - Easy to use, (Blackbox algorithm)
Current status for “European Option” type problems:

- Two kinds of schemes of order 2 (say Alg 1 [Victoir &N ’08] and Alg 2 [Ninomiya &N ’09]) Both work in practice.
- General extrapolation method for Alg 1 [Oshima, Teichmann and Veluscek ’09]
  Enables arbitraly order weak approximation

This talk is on:

- Higher order algorithms for barrier option pricing problem.
- New SCF to Asian option under Heston model.
References 1/3:


References 2/3:


References 3/3:

  International Workshop on Mathematical Finance “Topics on Leading-edge Numerical Procedures and Models” (16–18 Feb 2010, Tokyo)
The Problem:

Numerical calculation of $E[f(X(T, x))]$, where

$$X(t, x) = x + \sum_{j=0}^{d} \int_{0}^{t} V_j(X(s, x)) \circ dB^j(s)$$

$$V_j \in C^\infty_b(\mathbb{R}^N; \mathbb{R}^N)$$

$$B(t) = (B^0(t), B^1(t), \ldots, B^d(t)),$$

$$B^0(t) = t, \quad (B^1(t), \ldots, B^d(t)) : d\text{-dim Std. BM},$$

$\circ dB^j(s)$ : Stratonovich integral.
Two approaches:

- **PDE method**

  \[
  \frac{\partial u}{\partial t}(t, x) = Lu, \quad u(0, x) = f(x).
  \]

  where \( L = V_0 + (1/2) \sum_{i=1}^{d} V_i^2 \).

- **Probabilistic method — “Simulation”**

  **Step 1.** Discretize \( X(t, x) \) and obtain \( X^n(t, x) \).
  
  \( n = \# \{ \text{partitions of } [0, T] \} \)

  Euler–Maruyama, Higher order (Milstein, Kusuoka,..)

  **Step 2.** Integrate \( f(X^n(T, x)) \) over \( D(n) \)-dimensional domain \( [0, 1)^{D(n)} \)

  by MC, QMC, etc.

  \( D(n) \) depends on the discretization scheme

We consider only the probabilistic method here.
Simulation – MC and QMC

**Step. 2** of the simulation is the numerical integration:

\[
E \left[ f \left( X^n(T, x) \right) \right] = \int_{[0,1)^{D(n)}} F(a_1, \ldots, a_{D(n)}) \, da_1 \cdots da_{D(n)}
\]

Calc. RHS by using **MC** or **QMC**.
MC and QMC

\[ W : \text{r. v.}/(\Omega, \mathcal{F}, P), \quad M \in \mathbb{Z}_{>0} \]

\[ \text{MC}(W, M) := \frac{1}{M} \sum_{k=1}^{M} W_k, \quad \text{where } \{W_i\}_{i=1}^{M} : \text{iid s. t. } W_1 \sim W \]

\[ \text{QMC}(W, M) := \frac{1}{M} \sum_{k=1}^{M} W(\omega_k), \quad \{\omega_i\}_{i=1}^{\infty} : \text{deterministic sequence (LDS)} \]

\[ \text{MC}(W, M) : \quad \text{r. v.} \]

\[ \text{QMC}(W, M) \in \mathbb{R}. \]
Two types of approximation errors in simulation:

1. Discretization error

\[ |E[f(X(T, x))] - E[f(X^n(T, x))]| \]

2. Integration error

\[ |MC(f(X^n(T, x)), M)(\omega) - E[f(X^n(T, x))]| \]

or

\[ |QMC(f(X^n(T, x)), M) - E[f(X^n(T, x))]| \]
Two important remarks (1/2):
Integration error and MC By CLT,

$$\text{MC} \left( f \left( X^n(T, x) \right), M \right) \sim N \left( E \left[ f \left( X^n(T, x) \right) \right], \frac{\text{Var} \left[ f \left( X^n(T, x) \right) \right]}{M} \right).$$

When we proceed simulation

$$\text{Var} \left[ f(X(T, x)) \right] \approx \text{Var} \left[ f(X^n(T, x)) \right]$$

**Remark 1**
As long as we use MC, the number of sample points $M$ needed to attain the given accuracy is independent of $n$ and discretizing algorithm (Euler–Maruyama or Kusuoka etc.).
Two important remarks (2/2)

Integration error and QMC There exist sequences which satisfy

\[ \exists C_{f,n} > 0 \forall M \in \mathbb{Z}_{>0} \]

\[ \left| \text{QMC} \left( f \left( X^n(T, x) \right), M \right) - E \left[ f \left( X^n(T, x) \right) \right] \right| \leq C_{f,D(n)} \frac{(\log M)^{D(n)}}{M}. \]

Remark 2

In contrast to the MC case, the number of sample points \( M \) needed by the QMC to attain the given accuracy depends heavily on the dimension of integration \( D(n) \). Smaller the dimension, smaller number of samples are needed.

\[ D(n) = \begin{cases} 
  n \times d & \text{Euler–Maruyama,} \\
  n \times (d + 1) & \text{N. & Victoir,} \\
  2n \times d & \text{Ninomiya, & N.} 
\end{cases} \]
Order 1: Euler–Maruyama scheme

\[ X^{(EM),n}(0, x) = x, \]
\[ X^{(EM),n}\left(\frac{k + 1}{n}, x\right) = X^{(EM),n}\left(\frac{k}{n}, x\right) + \sqrt{\frac{T}{n}} \sum_{i=0}^{d} \tilde{V}_i \left( X^{(EM),n}\left(\frac{k}{n}, x\right) \right) Z_{k+1}^i, \]

where,

\[ \forall j \ Z_j^k = \begin{cases} \sqrt{T/n}, & \text{if } k = 0, \\ \text{iid. r. v. } \sim N(0, 1), & \text{if } k \in \{1, \ldots, d\}, \end{cases} \]
\[ \tilde{V}_k^i (y) = \begin{cases} V_0^i (y) + \frac{1}{2} \sum_j V_j V_j^i (y) & \text{if } k = 0, \\ V_k^i & \text{if } k \in \{1, \ldots, d\}. \end{cases} \]
Approx. Error of Euler–Maruyama scheme

\[ |E[f(X(T, x))] - E\left[f\left(X^{(EM), n}(T, x)\right)\right]| = O\left(n^{-1}\right) = O(\Delta t) \]

when \( f \): bdd. & measurable and \( \{V_i\}_{i=0}^d \): Unif. Hörmander Cond. [Bally & Talay '96][Kohatsu-Higa '00]

Euler–Maruyama scheme is an order 1 scheme.
Intuitive explanation of the scheme (1/3)

\((P^X_t f)(x) := E[f(X(t, x))], \quad f \in C^\infty_b(\mathbb{R}^N)\)

\[L := V_0 + \frac{1}{2} \sum_{j=1}^{d} V_j^2.\]
Intuitive explanation of the scheme (2/3)

Applying Ito formula repeatedly, we obtain

\[ E \left[ f(X(t, x)) \right] = (P^X_t f)(x) = f(x) + \int_0^t \left( P^X_{s_1} L f \right)(x) \, ds_1 \]

\[ = f(x) + \int_0^t \left\{ (Lf)(x) + \int_0^{s_1} \left( P^X_{s_2} L^2 f \right)(x) \, ds_2 \right\} \, ds_1 \]

\[ \vdots \]

\[ = \sum_{i=0}^n \left( \frac{tL} {i!} \right)^i f(x) + \frac{1} {n!} \int_0^t (t - s)^n \left( P^X_s L^{n+1} f \right)(x) \, ds. \]
Intuitive explanation of the scheme (3/3)

Observation: \[ \sum_{i=0}^{n} \left( \frac{(tL)^i}{i!} f \right)(x) \] gives a \( n \)th order approx. of \( E[f(X(t, x))] \).

Slogan: Construct a random variable \( \Xi \) s. t.

\[ E[\Xi] = \sum_{i=0}^{n} \left( \frac{(tL)^i}{i!} f \right)(x). \]
Free Lie Algebra

Non-commutative algebra (1)

\[ A := \{v_0, \ldots, v_d\} : \text{alphabet} \]

\[ A^* := \left( \bigcup_{k=1}^{\infty} A^k \right) \cup \{1\} : \text{free monoid on } A \]

For \( w = w_1 \cdots w_k \in A^*, \ (w_i \in A) \)

\[ |w| := k, \ |w| := |w| + \text{card } (\{1 \leq i \leq |w|; \ w_i = v_0\}) , \]

\[ A^*_m := \{ w \in A^* \ | |w| = m \}, \]

\[ A_{\leq m}^* := \{ w \in A^* \ | |w| \leq m \}. \]

**Concatenation product:**

For \( u = u_1 \cdots u_k, \ v = v_1 \cdots v_l \in A^* , \)

\[ u v := u_1 \cdots u_k v_1 \cdots v_l . \]
Non-commutative algebra (2)

\[ R\langle\langle A\rangle\rangle := \left\{ \sum_{w \in A^*} a_w w \mid a_w \in R \right\} : R\text{-algebra of formal series with basis } A^*, \]

\[ R\langle A \rangle := \left\{ \sum_{w \in A^*} a_w w \in R\langle\langle A\rangle\rangle \mid \exists k \in \mathbb{N} \text{ s.t. } a_w = 0 \text{ if } |w| \geq k \right\} \]

: free \( R \)-algebra with basis \( A^* \),

\( (P, w) := a_w, \text{ for } P = \sum_{w \in A^*} a_w w \in R\langle\langle A\rangle\rangle, \)

\( (PQ, w) := \sum_{uv = w} (P, u)(Q, v), \text{ for } P, Q \in R\langle\langle A\rangle\rangle, w \in A^*. \)
Non-commutative algebra (3)

- Projection and Truncation:

\[ j_m(P) := \sum_{\|w\| \leq m} (P, w)w \]

\[ P|_k := \sum_{\|w\| = k} (P, w)w \]

- Homogeneous component:

\[ \mathbb{R}\langle A \rangle_m := \{ P \in \mathbb{R}\langle A \rangle | (P, w) = 0 \text{ if } \|w\| \neq m \} \]
Free Lie algebra:

\[ [P, Q] := PQ - QP \quad \text{for } P, Q \in \mathbb{R}\langle\langle A\rangle\rangle, \]

\[ \mathcal{J}_A := \left\{ K \subset \text{sub } \mathbb{R}\text{-module} \mathbb{R}\langle A\rangle \bigg| A \subset K, [x, y] \in K \forall x, y \in K, \right\} \]

\[ \mathcal{L}_\mathbb{R}(A) := \bigcap_{K \in \mathcal{J}_A} K : \mathbb{R}\text{-coefficients free Lie algebra on } A \]

\[ \mathcal{L}_\mathbb{R}((A)) := \left\{ P \in \mathbb{R}\langle\langle A\rangle\rangle \text{ s.t. } P|_k \in \mathcal{L}_\mathbb{R}(A), \forall k \in \mathbb{N} \right\}. \]
Logarithm and exponential:
For $P \in \mathbb{R}\langle\langle A \rangle\rangle$ s.t. $(P, 1) = 0$

$$\exp(P) := 1 + \sum_{k=1}^{\infty} \frac{P^k}{k!}.$$ 

For $Q \in \mathbb{R}\langle\langle A \rangle\rangle$ s.t. $(Q, 1) = 1$

$$\log(Q) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(Q - 1)^k}{k}.$$ 

$\log(\exp(P)) = P$ and $\exp(\log(Q)) = Q$. 

S. Ninomiya
Higher-order weak approximation algorithms for SDEs
Hausdorff product:
For $z_1, z_2 \in L_\mathbb{R}((A))$,

$$z_1HZ_2 := \log(\exp(z_1) \exp(z_2)).$$

- $(z_1HZ_2)HZ_3 = \log(\exp(z_1) \exp(z_2) \exp(z_3)) = z_1H(z_2HZ_3) =: z_1HZ_2HZ_3$

- $z_2, z_1 \in L_\mathbb{R} \implies z_2HZ_1 \in L_\mathbb{R}((A))$
  $\therefore$ the Baker–Campbell–Hausdorff–Dynkin formula.
$\mathcal{L}_\mathbb{R}(A)$ and $\mathbb{R}\langle A \rangle$

Element of $\mathcal{L}_\mathbb{R} \iff$ Vector field generated by $V_0, \ldots, V_d$
Elements of $\mathbb{R}\langle A \rangle \iff$ Differential operator generated by $V_0, \ldots, V_d$
Resurgence and rescaling

- Resurgence operator $\Phi$:

$$
\Phi(1) := \text{id}, \quad \Phi(v_{i_1} \cdots v_{i_n}) := V_{i_1} \cdots V_{i_n}
$$

- Rescaling operator: For $s > 0$, $\psi_s : \mathbb{R} \langle \langle A \rangle \rangle \to \mathbb{R} \langle \langle A \rangle \rangle$ is defined as:

$$
\psi_s \left( \sum_{m=0}^{\infty} P_m \right) = \sum_{m=0}^{\infty} s^{m/2} P_m \quad \text{where } P_m \in \mathbb{R} \langle A \rangle_m.
$$

Example:

$$
\Phi \left( \psi_s \left( v_0 + \frac{1}{2} [v_1, v_2] \right) \right) = sV_0 + \frac{s}{2} [V_1, V_2]
$$
Notation: $\exp(V)(x)$

For a smooth vector field $V$, (i.e. $V \in C^\infty_b(\mathbb{R}^N; \mathbb{R}^N)$)

$$\exp(V)(x) := y(1)$$

where $y(t)$ is a solution to the ODE:

$$y(0) = x, \quad \frac{dy(t)}{dt} = V(y(t))$$

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Higher-order weak approximation algorithms for SDEs
Order $m$ Integration scheme: $\mathcal{IS}(m)$

\[ g \in \mathcal{IS}(m) \iff \begin{cases} & g : C_b^\infty(\mathbb{R}^N; \mathbb{R}^N) \rightarrow (\mathbb{R}^N \rightarrow \mathbb{R}^N), \\ & \exists C_m > 0 \text{ s.t. } \forall W \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N) \\ & \sup_{x \in \mathbb{R}^N} |g(W)(x) - \exp(W)(x)| \leq C_m (\|W\|_{C^{m+1}})^{m+1} \end{cases} \]

- $\mathcal{IS}(m) \equiv \text{“Set of order } m \text{ ODE solver”}$
- We need to integrate, for example:

\[ sV_0 + \frac{\sqrt{s}}{2}(V_1 + \cdots + V_d) \]

Not necessarily $s$-linear.
\( \mathcal{R}\langle\langle A\rangle\rangle, \mathcal{L}_\mathbb{R}(\langle A\rangle) \)-valued random variables

- Topology: \( \mathcal{R}\langle\langle A\rangle\rangle \approx \mathcal{R}^\infty \) – Direct product topology
- \( \mathcal{R}\langle\langle A\rangle\rangle \)-valued and \( \mathcal{L}_\mathbb{R}(\langle A\rangle) \)-valued probability theory can be considered.

(Ito formula, etc.)
Approximation theorem (1/2)

\( m \geq 1, \ M \geq 2, \) 
\( Z_1, \ldots, Z_M: \mathcal{L}_\mathbb{R}((A))\)-valued random variables s. t.

\[
Z_i = j_m Z_i \quad \text{for} \ i = 1, \ldots, M, \\
E [\|j_m Z_i\|_2] < \infty \quad \text{for} \ i = 1, \ldots, M, \\
E \left[ \exp \left( a \sum_{j=1}^{M} \| \Phi (\Psi_s(Z_j)) \|_{C^{m+1}} \right) \right] < \infty \quad \text{for any} \ a > 0.
\]
Approximation theorem (2/2)

\[ \forall p \in [1, \infty), \forall g_1, \ldots, g_M \in IS(m), \exists C_{m,M} > 0 \text{ s.t.} \]

\[ \left\| \sup_{x \in \mathbb{R}^N} \left| g_1 \left( \Phi \left( \Psi_s (Z_1) \right) \right) \circ \cdots \circ g_M \left( \Phi \left( \Psi_s (Z_M) \right) \right) (x) - \exp \left( \Phi \left( \Psi_s \left( j_m \left( Z_M H \cdots H Z_1 \right) \right) \right) \right) (x) \right\|_{L^p} \leq C_{m,M} s^{(m+1)/2} \]

for \[ \forall s \in (0, 1] \] where \( C_{m,M} \) depends only on \( m \) and \( M \).

\( f \circ g(x) := f \left( g(x) \right) \)
Cor. to the Approximation theorem

“$Q(s)$ gives $(m - 1)/2$-order weak approximation.”

Let $Q(s)$ for $s \in (0, 1]$ be

$$
(Q(s)f)(x) := E[f(g(\Phi(\psi_s(Z_1))) \circ \cdots \circ g(\Phi(\psi_s(Z_M))))(x)],
$$

where $f \in C^\infty_b(\mathbb{R}^N; \mathbb{R})$ and $g \in IS(m)$ then $\exists C > 0,$

$$
\|P_s f - Q(s)f\|_\infty \leq Cs^{(m+1)/2}\|\text{grad}(f)\|_\infty
$$

$P_s(f) := E[f(X(t, x))]$

(\textbf{Remark:} When $\{V_0, \ldots, V_d\}$ finitely generated.)
Algorithm 1

(Victoir–N. (2004))

\((\Lambda_i, Z_i)_{i \in \{1, \ldots, n\}} : 2n \text{ indep. r. v.},\)

\(\forall i \ P(\Lambda_i = \pm 1) = \frac{1}{2}, \ Z_i \sim N(0, I_d).\)

\(\{X_k^{(\text{Alg.1}), n}\}_{k=0, \ldots, n} : \text{a family of r. v. defined as:}\)

\[
X_0^{(\text{Alg.1}), n} := x,
\]

\[
X^{(\text{Alg.1}), n}_{(k+1)/n} :=
\begin{cases}
    \exp\left(\frac{V_0}{2n}\right)\exp\left(\frac{Z_k^1 V_1}{\sqrt{n}}\right) \cdots \exp\left(\frac{Z_k^d V_d}{\sqrt{n}}\right) \exp\left(\frac{V_0}{2n}\right) X_k^{(\text{Alg.1}), n} & \Lambda_k = +1, \\
    \exp\left(\frac{V_0}{2n}\right)\exp\left(\frac{Z_k^1 V_1}{\sqrt{n}}\right) \cdots \exp\left(\frac{Z_k^d V_d}{\sqrt{n}}\right) \exp\left(\frac{V_0}{2n}\right) X_k^{(\text{Alg.1}), n} & \Lambda_k = -1.
\end{cases}
\]
Extrapolations of Algorithm 1

- To an arbitrary order [Oshima, Teichmann and Veluscek ’09]
- To the 6th order [Fujiwara ’06]
General framework of Algorithm 2 (1)

- Remind the Slogan

- Find the Ξ written as:

\[
\log \Xi = Z_1 H Z_2 H \cdots H Z_M
\]

\[
Z_j = c_j v_0 + \sum_{i=1}^{d} S_{ij}^i v_i, \quad j \in \{1, \ldots, M\}
\]

where

\[
c_j \in \mathbb{R} \text{ s.t. } c_1 + \cdots + c_M = 1
\]

\[
E \left[ S_j^i S_j^{i'} \right] = R_{jj'} \delta_{ii'}
\]

\[
S_j^i \sim \mathcal{N}(0, R_{jj}) \quad (i \in \{1, \ldots, d\}, \ j \in \{1, \ldots, M\})
\]
General framework of Algorithm 2 (2)

The slogan is equivalent to:

\[ E[j_m (\exp(Z_1) \cdots \exp(Z_M))] = j_m \left( \exp \left( v_0 + \frac{1}{2} \sum_{i=1}^{d} v_i^2 \right) \right) \]  

(2)

The problem:

Find real numbers \( \{c_j\}_{j=1}^{M}, \{R_{ij}\}_{1 \leq i \leq j \leq M} \) that satisfy (2).

\[ \# \text{unknown vars} = \frac{1}{2} M(M + 3) \]
Main tools (Theorem 1):

For the LHS of (2)
If \( n^w(i) \) is odd for some \( i \in \{1, \ldots, d\} \), then \( C(w) = 0 \). If \( n^w(i) \) is even for every \( i \in \{1, \ldots, d\} \), then

\[
C(w) = \sum_{\vec{k} = (k_1, \ldots, k_M) \in \mathcal{K}_r(M)} \frac{1}{k_1! \cdots k_M!} \prod_{j=1}^{M} (c_j)^{N^w(0,j,\vec{k})} 
\]

\[
\times \prod_{p=1}^{d} \left( \sum_{\{d_{ij}\}^{1 \leq i \leq j \leq M} \in e(N^w(p,1,\vec{k}), \ldots, N^w(p,M,\vec{k}))} 2^{-\sum_{i=1}^{M} d_{ij}} \frac{\prod_{j=1}^{M} (N^w(p,j,\vec{k})!)}{\prod_{1 \leq i \leq j \leq M} (d_{ij}!)} \prod_{1 \leq i \leq j \leq M} R_{ij}^{d_{ij}} \right)^{(3)}
\]
Main tools (Theorem 2):

For the RHS of (2)

Let \( A^0 = \{v_0, v_1, v_2, \ldots, v_d \} \subset A^* \). Then

\[
\exp \left( v_0 + \frac{1}{2} \sum_{i=1}^{d} v_i^2 \right) = \sum_{w=w_1 \ldots w_l} \frac{1}{2^{|w|-l}!} w, \]

that is,

\[
C(w) = \begin{cases} 
\frac{1}{2^{|w|-l}!} & \text{if } w \in A^0 \\
0 & \text{otherwise.} 
\end{cases} \] (4)
Algebraic Relations

- Algebraic relations between \( \{c_j\}_{j=1}^M, \{R_{ij}\}_{1 \leq i \leq j \leq M} \)
- Using following 3 results, those relations are obtained.
An example of Algorithm 2 [Ninomiya–N. (2009)]

When \((m, M) = (5, 2)\),

\[
c_1 = \frac{\mp \sqrt{2} (2u - 1)}{2}, \quad c_2 = 1 \pm \frac{\sqrt{2} (2u - 1)}{2}
\]

\[
R_{22} = 1 + u \pm \sqrt{2 (2u - 1)}, \quad R_{12} = -u \mp \frac{\sqrt{2} (2u - 1)}{2}
\]

\[
R_{11} = u \quad \text{for some } u \geq 1/2.
\]

Becomes 1-dimensional ideal.
Partial results (1):

- The case $m \geq 6$:
  - $(m, M) = (7, 2)$: ideal becomes trivial  
    (i.e. $= \mathbb{C} [c_1, c_2, R_{11}, R_{12}, R_{13}]$)
  - $(m, M, d) = (7, 3, 1)$: ideal becomes trivial.
Partial results (2):

- The case \((m, M, d) = (7, 3, 2)\) and \(v_0\) free: The algebraic relation:

\[
\begin{align*}
14R_{22} + 24R_{13} - 13, & \quad 4R_{33} + 14R_{22} + 4R_{11} - 15, & \quad 36R_{33}^2 - 60R_{22} + 25, \\
-72R_{22}R_{33} + 68R_{33} + 24R_{23} + 22R_{22} - 17, & \quad -288R_{23}R_{33} - 8R_{33} + 264R_{23} - 46R_{22} + 77, \\
12R_{23} - 22R_{22} + 12R_{12} + 23, & \quad 8R_{33} + 216R_{22}R_{23} - 156R_{23} + 34R_{22} - 27 \\
-22R_{33} - 324R_{23}^2 - 192R_{23} + 46R_{22} - 57, & \quad 144R_{33}^2 - 64R_{33} + 168R_{23} + 46R_{22} + 7
\end{align*}
\]

Unfortunately, the solution is:

\[
R_{33} = \frac{5 \pm \sqrt{-11}}{12} \not\in \mathbb{R}
\]
Partial results (3):
The case \((m, M, d) = (7, 4, 3)\) and \(v_0\) free: Over 1-month symbolical calculation cannot find the answer.....
We have some more results (Theorem 3):

\[ A^* = \bigcup_{i=0}^{\infty} A^i, \; \mathbin{\lhd} : A^* \otimes A^* \to A^* : \text{Shuffle product}. \]

\((A^*, \mathbin{\lhd})\) becomes an algebra (shuffle algebra).

\[ C(\cdot) \]

\[ C : (A^*, \mathbin{\lhd}) \longrightarrow \mathbb{R}[c_j, R_{ij}, (1 \leq i \leq j \leq M)] \]

is ring homomorphism.

Cor.

\[ C \left( (0)^{n_0} \mathbin{\lhd} (11) \mathbin{\lhd} \cdots \mathbin{\lhd} (\ell \ell) \right) = C(0)^{n_0} C(11)^\ell \]
An example of Algorithm 2 [Ninomiya–N. (2005)]

\[
\begin{pmatrix}
Z_{i,k}^1 \\
Z_{i,k}^2
\end{pmatrix}_{i \in \{1, \ldots, d\}, j \in \{1, 2\}, k \in \{0, \ldots, n-1\}}
= \begin{pmatrix}
1/2 & 1/\sqrt{2} \\
1/2 & -1/\sqrt{2}
\end{pmatrix}
\begin{pmatrix}
\eta_{i,k}^1 \\
\eta_{i,k}^2
\end{pmatrix},
\quad \text{where } \eta_{i,k}^j \sim \text{i.i.d. } N(0, 1).
\]

\{X_{k}^{(\text{Alg.2)},n}\}_{k=0,\ldots,n} : \text{a family of r. v. defined as:}

\[
X_{0}^{(\text{Alg.2)},n} := x,
\]

\[
X_{(k+1)/n}^{(\text{Alg.2)},n} := 
\exp \left( \frac{1}{2n} V_0 + \sum_{i=1}^{d} \frac{Z_{i,k}^1}{\sqrt{n}} V_i \right) 
\exp \left( \frac{1}{2n} V_0 + \sum_{i=1}^{d} \frac{Z_{i,k}^M}{\sqrt{n}} V_i \right) X_{k/n}^{(\text{Alg.2}),n}
\]
Theorem
Both $X^{(\text{Alg.1}),n}$ and $X^{(\text{Alg.2}),n}$ are of order 2.

Recent result by Kusuoka
If $Q(s)$ is constructed by $X^{(\text{Alg.1}),n}$ or $X^{(\text{Alg.2}),n}$ then $\exists C > 0$ s.t.

$$(P_s f)(x) - (Q(s)f)(x) = Cs^{(m+1)/2} + O\left(s^{(m+3)/2}\right)$$
Higher order weak approximation algorithms
Recent developments
Our Higher-order scheme
Main result
The Algorithms
Numerical Example

Abstract

The weak approximation problem

The Algorithms

D(n)

D(n) : dimension of integration domain

\[
D(n) = \begin{cases} 
  n \times d & \text{Euler–Maruyama,} \\
  n \times (d + 1) & \text{Algorithm 1} \\
  n \times 2 \times d & \text{Algorithm 2}
\end{cases}
\]
The Runge–Kutta method

How to calc. $\exp(Z_i)(x)$?

- Lucky case: We can get exact form of $\exp(Z_i)(x)$. Often the case with Algorithm 1.
- Otherwise: Forced to proceed with numerical approximation.

Good news:

**Theorem [Ninomiya–N.]**

Classical order $m$ Runge–Kutta methods belongs to $IS(m)$
MC or QMC with Algorithms 1 and 2

- $W$: “a set of ODEs”-valued r. v. by Algorithm 1 or 2
- MC or QMC:
  - Draw a set of ODEs \( W(\omega) \) from \( W \) and obtain (something like) \( \exp(W(\omega))(x) \) numerically
  - Iterate the step above and calculate the average:

\[
\frac{1}{M} \sum_{i=1}^{M} \exp(W(\omega_i))(x)
\]
Advantages of the approach

- Free from symbolical calculation.
  - Calc. in group is easy.
    - Numerical ODE solver works (by the 2nd th’m)
  - Calc. in algebra is difficult.
    - Huge symbolical calc.
    - Simultaneous distributions of multiple integrations of BMs are not known. (except for the 2nd order.)

- Universal (applicable to non-commutative \( \{V_i\}_{i=0}^d \))
  Naïve “Ito-Taylor expansion with the Runge–Kutta” suffers from the non-commutativity.
Numerical Example:

Pricing of Asian option under the Heston model:

\[
Y_1(t, x) = x_1 + \int_0^t \mu Y_1(s, x) \, ds + \int_0^t Y_1(s, x) \sqrt{Y_2(s, x)} \, dW^1(s),
\]

\[
Y_2(t, x) = x_2 + \int_0^t \alpha (\theta - Y_2(s, x)) \, ds + \int_0^t \beta \sqrt{Y_2(s, x)} \, dW^2(s),
\]

\[
\langle W^1, W^2 \rangle_t = \rho t
\]

Asian Option:

\[
Y_3(t, x) = \int_0^t Y_1(s, x) \, ds, \quad \text{Payoff} = \max \left( \frac{Y_3(T, x)}{T} - K, 0 \right).
\]
Abstract

Higher order weak approximation algorithms
Recent developments
Appendix/Backup

The weak approximation problem
Our Higher-order scheme
Main result
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Numerical Example

Discretization Error

Discretization Error and Num. of Partitions

Euler-Maruyama
Euler-Maruyama + Romberg
Ninomiya-Victoir
Ninomiya-Victoir + Romberg
New Method
New Method + Romberg
O(1/n^2)
O(1/n^3)
5e-05
5e-06

Error vs Num. of Partitions

S. Ninomiya
Higher-order weak approximation algorithms for SDEs
Convergence Error from quasi-Monte Carlo and Monte Carlo

- QMC: New Method + Romberg, $1/\Delta t=2$
- QMC: Ninomiya-Victoir + Romberg, $1/\Delta t=4$
- QMC: New Method, $1/\Delta t=10$
- QMC: Ninomiya-Victoir, $1/\Delta t=12$
- QMC: Euler-Maruyama + Romberg, $1/\Delta t=16$
- MC: Euler-Maruyama

S. Ninomiya
Higher-order weak approximation algorithms for SDEs
Overall performance comparison:
#Partition, #Dim, #Sample, and CPU time required for $10^{-4}$ accuracy.

<table>
<thead>
<tr>
<th>Method</th>
<th>#Partition</th>
<th>#Dim</th>
<th>#Sample</th>
<th>CPU time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E-M + MC</td>
<td>2000</td>
<td>4000</td>
<td>$10^8$</td>
<td>$1.72 \times 10^5$</td>
</tr>
<tr>
<td>E-M + Extrpltn + QMC</td>
<td>16 + 8</td>
<td>48</td>
<td>$5 \times 10^6$</td>
<td>$1.27 \times 10^2$</td>
</tr>
<tr>
<td>N-V + QMC</td>
<td>12</td>
<td>36</td>
<td>$2 \times 10^5$</td>
<td>3.24</td>
</tr>
<tr>
<td>N-V + Extrpltn + QMC</td>
<td>4 + 2</td>
<td>18</td>
<td>$2 \times 10^5$</td>
<td>1.76</td>
</tr>
<tr>
<td>KNN + QMC</td>
<td>10</td>
<td>40</td>
<td>$2 \times 10^5$</td>
<td>3.4</td>
</tr>
<tr>
<td>KNN + Extrpltn + QMC</td>
<td>2 + 1</td>
<td>12</td>
<td>$2 \times 10^5$</td>
<td>1.2</td>
</tr>
</tbody>
</table>
Recent developments (1/2):

- Existence of the asymptotic expansion of the errors of Algorithms 1 and 2 [Kusuoka]
- Extrapolation of Algorithm 1
  - To order 6 [Fujiwara ’06]
  - To arbitrary order [Oshima, Teichmann and Veluscek ’09]
- SPDE case [Teichmann]
- Levy driven case [Tanaka and Kohatsu-Higa ’09]
Recent developments (2/2):

- Semi-closed form solutions to generalized SABR models by Algorithm 1 [Bayer, Friz and Loeffen ’10]
- Algorithms for Barrier-option pricing case [Kusuoka ’10][Kusuoka, Ninomiya and N. ’12]
- Semi-closed form solutions to Heston models by Algorithms 1 and 2.
Barrier option problem:
Calculate
\[
E \left[ f(X(t, x)), \min_{s \in [0,t]} X(s, t) > 0 \right]
\] (5)
umerically, where \(X(t, x)\) is a diffusion starting from \(x\).
The 1 dimensional case (i.e. $X : [0, 1] \times \mathbb{R} \times \Omega \to \mathbb{R}$)

Time horizon $= 1$.

\[
X(t, x) = x + \int_0^t V_0(X(s, x)) \, ds + \int_0^t dB(s)
\]

\[
(P_tf)(x) = E \left[ f(X(t, x)), \min_{s \in [0, t]} X(s, x) > 0 \right]
\]
killing function $k$

$$k : (0, 1] \times [0, \infty) \times \mathbb{R} \rightarrow [0, 1] \quad \text{measurable func.}$$

- Naive killing:
  $$k(s, x, y) = \begin{cases} 
  0, & \text{if } y > 0, \\
  1, & \text{otherwise}
  \end{cases}$$

- Standard killing:
  $$k(s, x, y) = \begin{cases} 
  \exp\left(-\frac{2xy}{s}\right), & \text{if } y > 0, \\
  1, & \text{otherwise}
  \end{cases}$$

- Improved standard killing:
  $$k(s, x, y) = \begin{cases} 
  \exp\left(\frac{2xy}{s}\right)k(s, x, y^{(2)}), & \text{if } y > 0,
  \end{cases}$$
Approximating operator $Q_s$:

$$(Q_s f)(x) := E \left[ f \left( F \left( X_{(n/N)}^N; x, s \right) \right) \left( 1 - k \left( s, x, F \left( X_{(n/N)}^N; x, s \right) \right) \right) \right]$$

where $F$ is the discretization scheme (ex. Euler-Maruyama, Alg-1, Alg-2).
Naive killing does not work 1

\[ V_0(x) = 1 + x \]

Discr Err: \( E[f(X(1)), X(t) > 0, t < 1], dX(t,x) = (1 + X(t,x))dt + dB(t), f: \text{digital call}, \text{naive killing}, g = 1, M = 100M \)

\[ V_0(x) = 1 + x \]
Naive killing does not work 2

\[ V_0(x) = \cos(x) + 3/2 \]

Discr Err: E[f(X(1)), X(t)>0, t<1], dX(t,x)=(\cos(X(t,x))+2/3)dt+dB(t), f: euro call, naive killing, g=1, M=100M
Standard killing

$$V_0(x) = 1 + x$$

Discr Err: $E[f(X(1)), X(t)>0, t<1], dX(t,x)=(1+X(t,x))dt+dB(t), f: digit. call, \gamma=1.0, Standard\ Killing$

S. Ninomiya
Higher-order weak approximation algorithms for SDEs
Standard killing

\[ V_0(x) = \cos(x) + \frac{3}{2} \]

Discr Err: \( E[\text{f}(X(1)), X(t) > 0, t < 1], \, dX(t, x) = (\cos(X(t, x)) + \frac{3}{2})dt + dB(t), \, \text{f: euro call}, g=1.0 \)
Standard killing

\[ V_0(x) = \cos(x) + \frac{3}{2} \]

Discr Err: \( E[f(X(1)), X(t) > 0, t < 1] \), \( dX(t) = (\cos(X(t)) + 3/2)dt + dB(t) \), f: euro call
Observation

- $Q(s)$ seems versatile and promising.
- Standard killing works.
- But, cannot see the “Improved” effect.
- Again, $\gamma$ paradox arises.
- Straightforward extension of the formula to multi-dimensional cases is possible.
Semi-closed form solution (SCF)  When all ODEs that arise in Alg. 1 have closed form solutions, the algorithm is called semi-closed form solution (SCF in the following).

Bayer-Friz-Loeffen ’10 [BFL10]

- SCFs to SABR and generalized SABR models.
- Transform the problem by adding constant drift:

\[ B^Q(t) = \gamma t + B^P(t) \quad \gamma \in \mathbb{R} \]

- SABR: One of the two important SV models in finance.
[Kubo–N. (2012)]

SCFs to the Asian option problem under Heston model by Alg 2.

**Heston model** The other important SV model in finance.

- The squared volatility process is of CIR type.

**Alg 2** ODEs are more complex than Alg 1 case.

**ODEs in Alg 1** \( \exp \left( \sqrt{\Delta t} \xi_i V_i \right) x \)

**ODEs in Alg 2** \( \exp \left( \frac{\Delta t}{2} V_0 + \sum_{i=1}^{d} \sqrt{\Delta t} \xi_i V_i \right) x \)
Asian Option under Heston model:

Numerical Calc. of $E \left[ \max \left( X^3(T)/T - K, 0 \right) \right]$ where

$X(t) = (X^1(t), X^2(t), X^3(t)), \quad X(0) = x \in \mathbb{R}^3$

$dX^1(t) = \mu X^1(t) \, dt + X^1(t) \sqrt{X^2(t)} \, dB^1(t)$

$dX^2(t) = \alpha (\theta - X^2(t)) \, dt + \beta \sqrt{X^2(t)} \left( \rho \, dB^1(t) + \sigma \, dB^2(t) \right)$

$dX^3(t) = X^1(t) \, dt$

$\rho^2 + \sigma^2 = 1, \, \rho \in (-1, 1)$ and the Feller condition:

$2\alpha \theta - \beta^2 > 0,$

which makes $X^2(t)$ strictly positive, is satisfied.
Derivation of the SCF (1/3)

Consider the following equivalent measure $Q$:

\[
\begin{align*}
g_1(t, X) &= -\frac{\sqrt{X^2(t)}}{2} - \frac{G}{\sqrt{X^2(t)}} \\
g_2(t, X) &= \frac{1}{\sigma} \left( H \sqrt{X^2(t)} + \frac{(I/\beta) + \rho G}{\sqrt{X^2(t)}} \right) \\
g(t, X) &= (g_1(t, X), g_2(t, X)), \quad B_Q(t) = \int_0^t g^\tau(s, X) \, ds + B(t) \\
L(t) &= \exp \left( -\int_0^t \sum_{j=1}^2 g^\tau_j(s, X) \, dB^j(s) - \frac{1}{2} \int_0^t |g^\tau(s, X)|^2 \, ds \right) \\
Q(A) &= E[1_A(L(T))]
\end{align*}
\]

where $G = \beta \rho / 4 - \mu$, $H = \rho / 2 - \alpha / \beta$, $I = \alpha \theta - \beta^2 / 4$ and $\tau$ is the 1st hitting time of $|g|$ to $M$. 
Derivation of the SCF(2/3)

Proceed the calculation considering as if \( g^\tau = g \), the problem is transformed as follows:

\[
V_0^Q(y) = .^t \left( 0, 0, y_1, C_1 y_2 + C_2 + C_3 / y_2 \right)
\]

\[
V_1(y) = .^t \left( y_1 \sqrt{y_2}, \beta \rho \sqrt{y_2}, 0, -g_1(t, y) \right)
\]

\[
V_2(y) = .^t \left( 0, \beta \sigma \sqrt{y_2}, 0, -g_2(t, y) \right)
\]

and

\[
dX(t) = V^Q(X(t)) \, dt + \sum_{j=1}^{2} V_j(X(t)) \circ dB^j_Q(t)
\]

where \( C_1 = 1/8 + H^2/(2\sigma^2) \), \( C_2 = G/2 - \alpha/4 + H(I/\beta + \rho G)/\sigma^2 \), \( C_3 = (G^2 - I/2 + (I/\beta + \rho G)^2/\sigma^2)/2 \) and \( X^4(t) := \log L(t) \).
Derivation of the SCF(3/3)

At last we get:

\[
\exp\left(t\left(\frac{s}{2}V_0^Q + \sqrt{s}Z_1 V_1 + \sqrt{s}Z_2 V_2\right)\right) x
\]

\[
= \begin{pmatrix}
    x_1 \exp\left(\sqrt{s}Z_1 \left(\frac{Kt^2}{2} + \sqrt{x_2}t\right)\right) \\
    (Kt + \sqrt{x_2})^2 \\
    x_3 + \frac{sx_1}{2} e^{-\sqrt{s}Z_1(x_2/2K)} \int_{\sqrt{x_2}/K}^{t+\sqrt{x_2}/K} \exp\left(\sqrt{s}Z_1 \frac{K}{2} u^2\right) du \\
    x_4 + A(t, \sqrt{x_2}, 1/ \sqrt{x_2}, 1/(Kt + \sqrt{x_2})) + \log\left(\frac{Kt}{\sqrt{x_2}} + 1\right)
\end{pmatrix}
\]

where \( K = \frac{\beta \sqrt{x_2}}{2} (\rho Z_1 + \sigma Z_2) \) and \( A(u, v, w, z) \) is a polynomial of \( u, v, w \) and \( z \).
On the new SCFs to Heston model

- Two new SCFs to Asian derivatives under Heston model
- Another simulation method for CIR processes
- Need error estimation w.r.t $M$
- Work well in practical examples
Why it works well? (in spite of the cheating)

Rough calculation of $P(X^2(t) \leq 0)$: When $\theta = 0.09$, $\alpha = 2.0$, $\beta = 0.1$, $\mu = 0.05$, $\rho = -1/2$, $x_2 = 0.09$ and $T = 1$,

$$\text{Var}(X^2(1)) = 0.25 \times 10^{-2} \quad \text{and} \quad \sqrt{x_2}/\text{Var}(X^2(1)) = 6.0.$$  

Then

$$P(X^2(T) \leq 0) = \frac{1}{\sqrt{2\pi}} \int_{6.0}^{\infty} e^{-u^2/2} \, du \approx 2.2 \times 10^{-17}$$
Both approaches are necessary because:

- PDE approach works only when
  - $L$ is elliptic,
  - dimension is small.
- Simulation is the last resort but (people believes) very time consuming.

This presentation focuses on “Simulation.”