Geometric vs non-geometric rough paths

David Kelly    Martin Hairer

Mathematics Institute
University of Warwick
Coventry UK CV4 7AL

dtbkelly@gmail.com

September 27, 2012

Data Assimilation 2012, Oxford.
The problem

We are interested in equations of the form

\[ dY_t = \sum_i f_i(Y_t) dX_t^i , \]

where \( X : [0, T] \rightarrow \mathcal{V} \) is path with some Hölder exponent \( \gamma \in (0, 1) \), \( Y : [0, T] \rightarrow \mathcal{U} \) and \( f_i : \mathcal{U} \rightarrow \mathcal{U} \) are smooth vector fields.

The theory of rough paths (Lyons) tells us that we should think of the equation as

\[ dY_t = \sum_i f_i(Y_t) d\mathbf{X}_t , \quad (†) \]

where \( \mathbf{X} \) is an object containing \( X \) as well as information about the iterated integrals of \( X \). We call \( \mathbf{X} \) a rough path above \( X \).
The problem

We are interested in equations of the form

$$dY_t = \sum_i f_i(Y_t)dX^i_t,$$

where $X : [0, T] \rightarrow V$ is path with some Hölder exponent $\gamma \in (0, 1)$, $Y : [0, T] \rightarrow U$ and $f_i : U \rightarrow U$ are smooth vector fields.

The theory of rough paths (Lyons) tells us that we should think of the equation as

$$dY_t = \sum_i f_i(Y_t)dX_t^i,$$  \hspace{1cm} (†)

where $X$ is an object containing $X$ as well as information about the iterated integrals of $X$. We call $X$ a rough path above $X$. 
What is a rough path

- $X$ lives in the tensor product space $X : [0, T] \to V \oplus V \otimes 2 \oplus \cdots \oplus V \otimes N$ where $N$ is the largest integer such that $N\gamma \leq 1$.
- $X$ lives above $X$ in that $\langle X_t, e_i \rangle = X^i_t$.
- The tensor components encode the iterated integrals of $X$

$$\langle X_t, e_i \otimes e_j \rangle = \int_0^t \int_0^r dX^i_r dX^j_r$$

and

$$\langle X_t, e_i \otimes e_j \otimes e_k \rangle = \int_0^t \int_0^r \int_0^u dX^i_v dX^j_u dX^k_r$$

- $X$ is usually assumed to be geometric, which means that the integrals obey the “usual laws of calculus”.

What is a rough path

- \( X \) lives in the tensor product space \( X : [0, T] \rightarrow V \oplus V \otimes 2 \oplus \cdots \oplus V \otimes N \)
  where \( N \) is the largest integer such that \( N \gamma \leq 1 \).
- \( X \) lives above \( X \) in that \( \langle X_t, e_i \rangle = X^i_t \).
- The tensor components encode the **iterated integrals** of \( X \)
  \[
  \langle X_t, e_i \otimes e_j \rangle = \int_0^t \int_0^r dX^i_r dX^j_r
  \]
  and  \( \langle X_t, e_i \otimes e_j \otimes e_k \rangle = \int_0^t \int_0^r \int_0^u dX^i_v dX^j_u dX^k_r \)
- \( X \) is usually assumed to be **geometric**, which means that the
  integrals obey the “usual laws of calculus”.
What is a rough path

- \textbf{X} lives in the tensor product space \( \textbf{X} : [0, T] \to V \oplus V \otimes^2 \oplus \cdots \oplus V \otimes^N \)
  where \( N \) is the largest integer such that \( N \gamma \leq 1 \).
- \textbf{X} lives above \( \textbf{X} \) in that \( \langle \textbf{X}_t, e_i \rangle = X^i_t \).
- The tensor components encode the \textbf{iterated integrals} of \( \textbf{X} \)
  \[
  \langle \textbf{X}_t, e_i \otimes e_j \rangle = \int_0^t \int_0^r dX^i_r dX^j_r \\
  \text{and} \quad \langle \textbf{X}_t, e_i \otimes e_j \otimes e_k \rangle = \int_0^t \int_0^r \int_0^u dX^i_v dX^j_u dX^k_r 
  \]
- \( \textbf{X} \) is usually assumed to be \textbf{geometric}, which means that the integrals obey the “usual laws of calculus”.
What is a rough path

• $\mathbf{X}$ lives in the tensor product space $\mathbf{X} : [0, T] \to V \oplus V \otimes 2 \oplus \cdots \oplus V \otimes N$
  where $N$ is the largest integer such that $N \gamma \leq 1$.

• $\mathbf{X}$ lives above $\mathbf{X}$ in that $\langle \mathbf{X}_t, e_i \rangle = X^i_t$.

• The tensor components encode the **iterated integrals** of $\mathbf{X}$

\[
\langle \mathbf{X}_t, e_i \otimes e_j \rangle = \int_0^t \int_0^r d\mathbf{X}_r^i d\mathbf{X}_r^j
\]

and
\[
\langle \mathbf{X}_t, e_i \otimes e_j \otimes e_k \rangle = \int_0^t \int_0^r \int_0^u d\mathbf{X}_r^i d\mathbf{X}_u^j d\mathbf{X}_r^k
\]

• $\mathbf{X}$ is usually assumed to be **geometric**, which means that the integrals obey the “usual laws of calculus”.

Non-geometric rough paths

What if the integrals in equations like (†) don’t obey the usual laws of calculus?

Eg. Riemann-sum integrals for non-semimartingales (Burdzy, Swanson), Regularised integrals (Russo, Vallois).

This still fits into the framework of rough paths, but we need to add a few more components to $X$. 
Non-geometric rough paths

What if the integrals in equations like \( \text{\dag} \) don’t obey the usual laws of calculus?

Eg. Riemann-sum integrals for non-semimartingales (Burdzy, Swanson), Regularised integrals (Russo, Vallois).

This still fits into the framework of rough paths, but we need to add a few more components to \( X \).
Non-geometric rough paths

Instead of tensors, the components of $X$ are indexed by labelled trees

\[
\bullet i, \bullet j, \bullet k, \bullet_{ik}, \ldots
\]

with the same labels used to index the basis of $V$. 
Non-geometric rough paths

Instead of tensors, the components of $X$ are indexed by labelled trees

\[ \bullet_i, \bullet_{ij}, \bullet_{ijk}, \bullet_{ijkl}, \ldots \]

with the same labels used to index the basis of $V$. And we have

\[
\langle X_t, \bullet_i \rangle = X_t^i, \quad \langle X_t, \bullet_{ij} \rangle = \int_0^t \int_0^r dX^i_u dX^j_r
\]

\[
\langle X_t, \bullet_{ijk} \rangle = \int_0^t \int_0^r \int_0^u dX^i_v dX^j_u dX^k_r, \quad \langle X_t, \bullet_{ikj} \rangle = \int_0^t X_r^i X_r^j dX_r^k
\]

The object $X$ is known as a branched rough path (Gubinelli).
Removing the branches

**Theorem (MH, DK)**

*Every branched rough path can be encoded in a geometric rough path.*

For some $X$ with a branched rough path $X$ above it. There exists a path $\bar{X} = (X, \ldots)$ with a geometric rough path $\bar{X}$ above it, satisfying

$$\langle X_t, \tau \rangle = \langle \bar{X}_t, \psi(\tau) \rangle$$

for every tree $\tau$.

The components of $\bar{X}$ above $X$ can be any geometric rough path.
Removing the branches

Theorem (MH, DK)

Every branched rough path can be encoded in a geometric rough path.

For some $X$ with a branched rough path $X$ above it, there exists a path $\bar{X} = (X, \ldots)$ with a geometric rough path $\bar{X}$ above it, satisfying

$$\langle X_t, \tau \rangle = \langle \bar{X}_t, \psi(\tau) \rangle$$

for every tree $\tau$.

The components of $\bar{X}$ above $X$ can be any geometric rough path.
Removing the branches

**Theorem (MH,DK)**

*Every branched rough path can be encoded in a geometric rough path.*

For some $X$ with a branched rough path $X$ above it. There exists a path $\bar{X} = (X, \ldots)$

with a geometric rough path $\bar{X}$ above it, satisfying

$$\langle X_t, \tau \rangle = \langle \bar{X}_t, \psi(\tau) \rangle.$$

for every tree $\tau$.

The components of $\bar{X}$ above $X$ can be any geometric rough path.
Removing the branches

**Theorem (MH, DK)**

Every branched rough path can be encoded in a geometric rough path.

For some $X$ with a branched rough path $\bar{X}$ above it. There exists a path

\[ \bar{X} = (X, \ldots) \]

with a geometric rough path $\bar{X}$ above it, satisfying

\[ \langle X_t, \tau \rangle = \langle \bar{X}_t, \psi(\tau) \rangle. \]

for every tree $\tau$.

The components of $\bar{X}$ above $X$ can be any geometric rough path.
Removing the branches

**Theorem (MH, DK)**

*Every branched rough path can be encoded in a geometric rough path.*

For some $X$ with a branched rough path $X$ above it. There exists a path

\[
\begin{align*}
X & \quad \psi \quad \bar{X} \\
\bar{X} & \quad \bar{X}
\end{align*}
\]

with a geometric rough path $\bar{X}$ above it, satisfying

\[
\langle X_t, \tau \rangle = \langle \bar{X}_t, \psi(\tau) \rangle.
\]

for every tree $\tau$.

The components of $\bar{X}$ above $X$ can be any geometric rough path.
Consequences for rough DEs

Corollary (generalised Itô-Stratonovich correction formula)

$Y$ is a solution to

$$dY_t = \sum_i f_i(Y_t)dX^i_t \quad (\text{driven by } X)$$

if and only if $Y$ is a solution to the rough DE

$$dY = \sum_i f_i(Y) \circ dX^i_t + \sum_{\tau} \bar{f}_\tau(Y_t) \circ d\bar{X}^\tau_t \quad (\text{driven by } \bar{X}),$$

where $\bar{X}$ is the geometric rough path derived above.

\textbf{nb.} Should really be called any non-geometric integral - any geometric integral correction formula.
Consequences for rough DEs

Corollary (generalised Itô-Stratonovich correction formula)

\( Y \) is a solution to

\[
dY_t = \sum_i f_i(Y_t) dX_t^i \quad (\text{driven by } X)
\]

if and only if \( Y \) is a solution to the rough DE

\[
dY = \sum_i f_i(Y) \circ dX_t^i + \sum_{\tau} \bar{f}_{\tau}(Y_t) \circ d\bar{X}_t^\tau \quad (\text{driven by } \bar{X}),
\]

where \( \bar{X} \) is the geometric rough path derived above.

\textbf{nb.} Should really be called \textit{any non-geometric integral - any geometric integral} correction formula.