THAT COSTS WHAT!

PROBABILISTIC LEARNING FOR VOLATILITY & OPTIONS
VOLATILITY & OPTIONS
S&P 500 index

S&P 500 [USD]

Today: $1115.1
Volatility: measure of variability of returns
Volatility: measure of variability of returns
Historical vs. forward-looking volatility:

Stock vs. option prices!
Call option: The right to buy a stock at future maturity date for strike-price agreed today (sell—put)
At maturity: \( \text{payoff} = (\text{price} - \text{strike})^+ \)
Black-Scholes option pricing

• Stock price under real-world measure $P$

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

• Volatility parameter is a constant
Black-Scholes option pricing

- Stock price under real-world measure $P$

\[ dS_t = \mu S_t dt + \sigma S_t dW_t \]

- Volatility parameter is a constant

- Invariant under change to pricing measure $Q \sim P$

\[ dS_t = r S_t dt + \sigma S_t d\tilde{W}_t \]

\[ C_t(T, K) = \mathbb{E}^Q \left[ e^{-rT} (S_T - K)^+ | S_t = s \right] \]
Black-Scholes option pricing

- Under Black-Scholes, historical volatility is the same as option-implied, forward-looking volatility — a constant parameter
EUROPEAN OPTION MARKET
S&P 500 call options

473 quoted strike-maturity points (2010-01-04)
S&P 500 call options

Bid-ask prices
Implied volatility: market’s estimate of future volatility
Historical volatility

MODEL

European options

Implied volatility
Fwd./market bet

PRICING/HEDGING/RISK

Derivatives/exotics
OTC market
Local Volatility Model
Local volatility model

- Stock price, $Q$-dynamics

\[ dS_t = rS_t dt + \sigma(t, S_t)S_t dW_t, \quad t \in [0, T^*] \]

- Local volatility function—our modelling target

\[ \sigma : [0, T^*] \times \mathbb{R}_+ \to \mathbb{R} \]
Local volatility model

Risk-neutral pricing: function-to-function mapping

\[ \sigma(\cdot, \cdot) \mapsto C(\cdot, \cdot) \]
Why local volatility?

• A model can reproduce a set of option prices if within range attainable by model.

• For local volatility, *any set* attainable as long as consistent in the sense of no static arbitrage.

• *Uniqueness* in the limit of infinite data.

• In practice, need calibration approach to fit model to observed data.
Why local volatility?

[Graph showing the relationship between strike, maturity, local volatility, and model price with mean square error.]
Probabilistic Approach
Bayesian framework

- We take a *Gaussian process* to define a functional prior distribution over local volatility

- Likelihood: data is *noisy observations* of true fair price

- Infer *posterior distribution* over local volatility from observed data
Inference

• Calibration ~ sampling from posterior

• Provides “best” point-estimate(s) in terms of model-to-market error and notion of uncertainty in estimate

• Rich probabilistic representation of calibrated model
Markov chain Monte Carlo
Posterior local volatility

![Graph depicting posterior local volatility with strike in USD and maturity in years. The graph shows a 3D plot with local volatility on the y-axis, strike on the x-axis, and maturity on the z-axis. There are also 2D plots showing local volatility with strike and maturity. The graph includes a note indicating maturity: 0.13 year.]
Posterior local volatility

strike [USD]
1000
1500

maturity [years]
0.5
1.0
1.5
2.0
2.5

local volatility
0.0
0.2
0.4
0.6
0.8

±2SD
MAP
data loc.
maturity: 3 year

strike
600
800
1000
1200
1400
1600
1800

local volatility
0.0
0.2
0.4
0.6
0.8
Posterior uncertainty

1. Nonlinearity of pricing operator
   - Price sensitivity to changes in local vol across $T$ and $K$

2. Availability of observed data
   - Density of prices across $T$ and $K$
Posterior option prices

- Feed local volatility back into model for posterior over prices and implied volatility
Posterior option prices

- Call price distribution
- Implied volatility distribution

Maturity: 0.13 year
Posterior option prices

The graphs depict the posterior option prices with maturity 0.48 year. The left graph shows the call prices as a function of the strike, with the data points indicating the MAP estimate and the shaded area representing the ±2SD confidence interval. The right graph illustrates the implied volatility similarly. Both graphs demonstrate the decrease in option prices and implied volatility with increasing strike.
Posterior option prices

maturity: 0.99 year

strike

call price

±2SD
MAP
data

implied volatility

±2SD
MAP
data

maturity: 0.99 year

strike
Posterior option prices

![Call Price Graph](left)

![Implied Volatility Graph](right)

- **Maturity:** 3 years
- **Call Price** graph shows data for different strikes, with a downward trend.
- **Implied Volatility** graph also shows data for different strikes, with a downward trend.
- Both graphs include ±2SD and MAP data markers.

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**Observations:**

- The call price graph indicates a decrease as the strike price increases.
- The implied volatility graph also shows a decrease as the strike price increases.
- The data markers represent ±2SD and MAP data points.
PART 2:
VOLATILITY DYNAMICS
S&P 500 option market
Volatility dynamics

- Local volatility provides good static fit of option market; need to re-calibrate on regular basis
- Our framework provides way of inferencing dynamical behaviour by introducing temporal dependency in the Gaussian process prior
- *Posterior* over local volatility (call prices, implied vols etc.) with dependency across time
Calibration
Prediction

- The Gaussian process readily provides *predictive distribution* over local volatility (prices, implied vol etc.)

- Since we model temporal dependency, may predict (local, prices, implied) volatility

- We use the posterior to make *one-week ahead* predictions
Prediction
Prediction
Realised volatility: historical measure from returns
VIX-index: 30-day forward volatility measure implied by market—1-week ahead predictions
Concluding remarks

• We present an approach for nonparametric modelling of local volatility

• The approach is probabilistic, gives notion of uncertainty

• Using Gaussian processes, it is flexible and scales with data
Concluding remarks

• Connection to “classical” inverse calibration with regularised optimisation, but gives explicit link to probabilistic framework

• Including time dimension in input variable, gives means of studying temporal behaviour of (local) volatility over time, as well as for predicting future surfaces

• Computational methods (MCMC) expensive
Thank you for your attention!