The multivariate lack-of-memory property: Analytical characterizations & applications to mathematical finance

Oxford, 13-Nov-2012

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Univariate LOM property

Univariate LOM:

\[ P(\tau > t + s \mid \tau > t) = P(\tau > s), \quad \forall t, s \in \mathcal{T} \]

(i) \( \mathcal{T} = \mathbb{R}_+ \iff \tau \sim \mathcal{E}(\lambda), \lambda > 0 \)

(ii) \( \mathcal{T} = \mathbb{N}_0 \iff \tau \sim \mathcal{G}(1 - p), \; p \in (0, 1) \)
Univariate LOM property

Univariate LOM:

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(i) \( \mathcal{T} = \mathbb{R}_+ \iff \tau \sim \mathcal{E}(\lambda), \lambda > 0 \)

(ii) \( \mathcal{T} = \mathbb{N}_0 \iff \tau \sim \mathcal{G}(1 - p), p \in (0,1) \)

• The “base distributions” in reliability theory

• Connection:

\[ \tau \sim \mathcal{G}(1 - p) \iff \tau \overset{d}{=} \lceil \tilde{\tau} \rceil, \tilde{\tau} \sim \mathcal{E}(-\log(p)) \]
Construction of default times $\tau$

**Exponential case:** $\tau \sim \mathcal{E}(\lambda)$

$$\tau \overset{d}{=} \inf\left\{ t \geq 0 : \int_{0}^{t} \lambda \, ds \geq E \right\}, \quad E \sim \mathcal{E}(1)$$
Construction of default times $\tau$

**Exponential case:** $\tau \sim \mathcal{E}(\lambda)$

$$
\tau \overset{d}{=} \inf\{t \geq 0 : \int_0^t \lambda ds \geq E\}, \quad E \sim \mathcal{E}(1)
$$

**Generalizations beyond LOM:**

$$
\tau := \inf\{t \geq 0 : \int_0^t \lambda(s) ds \geq E\}, \quad \text{(deterministic intensity)}
$$

$$
\tau := \inf\{t \geq 0 : \int_0^t \lambda_s ds \geq E\}, \quad \text{(stochastic intensity)}
$$

$$
\tau := \inf\{t \geq 0 : \Lambda_t \geq E\}, \quad \text{(stochastic cumulative hazard)}
$$

(c) Mai, Scherer, Shenkman
Construction of default times $\tau$

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$$\tau := \inf\{t \geq 0 : \Lambda_t \geq E\}, \quad \text{(stochastic cumulative hazard)}$$

**Geometric case:** $\tau \sim \mathcal{G}(1 - p)$

$$\tau := \min\{n \in \mathbb{N} : \sum_{j=1}^n \ldots \geq E\}, \quad (3 \text{ cases, as above})$$

(c) Mai, Scherer, Shenkman
Multivariate LOM

$(\tau_1, \ldots, \tau_d)$ satisfies LOM if $\forall s_1, \ldots, s_d, t \in \mathcal{T}$ and arbitrary $1 \leq i_1, \ldots, i_k \leq d$:

$$P(\tau_{i_1} > s_{i_1} + t, \ldots, \tau_{i_k} > s_{i_k} + t \mid \tau_{i_1} > t, \ldots, \tau_{i_k} > t) = P(\tau_{i_1} > s_{i_1}, \ldots, \tau_{i_k} > s_{i_k})$$
Multivariate LOM

$(\tau_1, \ldots, \tau_d)$ satisfies LOM if $\forall s_1, \ldots, s_d, t \in T$ and arbitrary $1 \leq i_1, \ldots, i_k \leq d$:

$$\mathbb{P}(\tau_{i_1} > s_{i_1} + t, \ldots, \tau_{i_k} > s_{i_k} + t \mid \tau_{i_1} > t, \ldots, \tau_{i_k} > t) = \mathbb{P}(\tau_{i_1} > s_{i_1}, \ldots, \tau_{i_k} > s_{i_k})$$

Questions:

1. Canonical stochastic construction of $(\tau_1, \ldots, \tau_d)$?

2. Analytical characterization:

   $\text{LOM } (T = \mathbb{R}_+) \iff \mathbb{P}(\tau_1 > t_1, \ldots, \tau_d > t_d) = ?$
   
   $\text{LOM } (T = \mathbb{N}_0) \iff \mathbb{P}(\tau_1 > t_1, \ldots, \tau_d > t_d) = ?$

3. Stylized facts for dependent defaults?

4. Tractable parametric families?

5. Interesting portfolio default models?
Marshall–Olkin distribution

(1) Stochastic model by [Marshall, Olkin (1967)]:

- \( \lambda_I \geq 0, \emptyset \neq I \subseteq \{1, \ldots, d\} \), with \( \sum_{I:k \in I} \lambda_I > 0 \) for \( k = 1, \ldots, d \)
- \( E_I \sim \mathcal{E}(\lambda_I) \) independent
- \( (\tau_1, \ldots, \tau_d) \) defined by
  \[
  \tau_k := \min\{E_I : k \in I\}
  \]
Marshall–Olkin distribution

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- $(\tau_1, \ldots, \tau_d)$ defined by
  \[
  \tau_k := \min\{E_I : k \in I\}
  \]

(2) Then $(\tau_1, \ldots, \tau_d)$ has support $[0, \infty)^d$ with survival function
  \[
  \mathbb{P}(\tau_1 > t_1, \ldots, \tau_d > t_d) = \exp\left(-\sum_{\emptyset \neq I \subseteq \{1, \ldots, d\}} \lambda_I \max_{i\in I}\{t_i\}\right)
  \]

Theorem: This is the only law on $[0, \infty)^d$ that satisfies LOM
Narrow-sense multivariate geometric distribution $G^N$

(1) We “discretize” the Marshall–Olkin model by applying $\lceil . \rceil$ componentwise:

- $p_I \in [0, 1], \emptyset \neq I \subseteq \{1, \ldots, d\}$, with $\prod_{I: k \in I} p_I < 1$ for $k = 1, \ldots, d$
- $G_I \sim G(1 - p_I)$ independent
- $(\tau_1, \ldots, \tau_d)$ defined by
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  \tau_k := \min\{G_I : k \in I\}
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(2) Then $(\tau_1, \ldots, \tau_d)$ has support $\mathbb{N}^d$ with discrete survival function

$$\mathbb{P}(\tau_1 > n_1, \ldots, \tau_d > n_d) = \prod_{\emptyset \neq I \subseteq \{1, \ldots, d\}} p_I^{\max_{i \in I} \{n_i\}}$$

It satisfies LOM, but is not canonical, i.e. there are more LOM-laws on $\mathbb{N}^d$

(c) Mai, Scherer, Shenkman
Wide-sense multivariate Geometric distribution $\mathcal{G}^W$

(1) Stochastic model by [Arnold (1975)]:

- $\tilde{p}_I \in [0, 1]$, $I \subseteq \{1, \ldots, d\}$, with $\sum_I \tilde{p}_I = 1$ and $\sum_{I: k \notin I} \tilde{p}_I < 1$ for $k = 1, \ldots, d$
- Repeatedly run experiment with outcomes $I \subseteq \{1, \ldots, d\}$
- $G_I$: number of the experiment which first yielded $I$
- $(\tau_1, \ldots, \tau_d)$ defined by

$$\tau_k := \min\{G_I : k \in I\}$$
Wide-sense multivariate Geometric distribution $G^W$

(1) Stochastic model by [Arnold (1975)]:

- $\tilde{p}_I \in [0, 1], I \subseteq \{1, \ldots, d\}$, with $\sum I \tilde{p}_I = 1$ and $\sum I: k \notin I \tilde{p}_I < 1$ for $k = 1, \ldots, d$
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  $$\tau_k := \min\{G_I : k \in I\}$$

(2) Then $(\tau_1, \ldots, \tau_d)$ has support $\mathbb{N}^d$ with discrete survival function

$$\mathbb{P}(\tau_1 > n_1, \ldots, \tau_d > n_d) = \prod_{k=1}^d \left( \sum_{\pi_n(i) \notin I \forall i = k, \ldots, d} \tilde{p}_I \right)^{n(k) - n(k-1)},$$

where $\pi_n: \{1, \ldots, d\} \to \{1, \ldots, d\}$ is a permutation depending on $n = (n_1, \ldots, n_d)$ s.t. $n_{\pi(1)} \leq n_{\pi(2)} \leq \ldots \leq n_{\pi(d)}$, and $0 =: n(0) \leq n(1) \leq \ldots \leq n(d)$

**Theorem:** This is the only law on $\mathbb{N}^d$ satisfying discrete LOM
Difference between $\mathcal{G}^N$ and $\mathcal{G}^W$

- $\mathcal{G}^N$ is constructed “at once”:
  - Simulate all (possible) shock times $G_I$
  - Determine the extinction times
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- $\mathcal{G}^W$ is constructed “repeatedly”:
  - Simulate extinctions of the first period
  - Simulate extinctions of the second period
  - ...
  - Until all components are extinct

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Difference between $G^N$ and $G^W$

- $G^N$ is constructed “at once”:
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- $G^W$ is constructed “repeatedly”:
  - Simulate extinctions of the first period
  - Simulate extinctions of the second period
  - ...  
  - Until all components are extinct

- $G^W$ allows for negative correlations, $G^N$ doesn’t!
Reality check 1/2

(3) Properties of $(\tau_1, \ldots, \tau_d)$ and interpretation:

- Singular component $\Rightarrow$ **catastrophic events / joint defaults**
- Positive lower tail dependence $\Rightarrow$ **joint early defaults**
- LOM $\Rightarrow$ **no contagion effects**

**Tractability:**

- $2^d - 1$ parameters $\Rightarrow$ **impossible to calibrate**
- Simulation requires exponential effort in $d$ $\Rightarrow$ **too slow**
Reality check 2/2

(3) **Required when working with (large) credit portfolios:**

(a) Tractable subfamilies (fewer parameters)

(b) A convenient stochastic representation for simulation

(c) Generalizations beyond LOM

(d) The distribution of the portfolio loss $L_t$, since we need

$$
\mathbb{E}[f(L_t)] = \int_{[0,1]} f(x) \mathbb{P}(L_t \in dx), \quad f \text{ non-linear}, \quad L_t = \frac{1}{d} \sum_{k=1}^{d} 1_{\{\tau_k \leq t\}}
$$

(e) The portfolio’s dimension should be flexible, since:
   - New credits are added to the portfolio
   - Credits are removed (repaid, defaulted, etc.) from the portfolio
CIID subfamily

(4) **Idea:** Identify conditionally iid (CIID) subfamily and apply LHP approximation
CIID subfamily

(4) **Idea:** Identify conditionally iid (CIID) subfamily and apply LHP approximation

**[De Finetti (1937)] Theorem:** A sequence \( \{\tau_k\}_{k \in \mathbb{N}} \) on \((\Omega, \mathcal{F}, \mathbb{P})\) is exchangeable \(\iff\) it is conditionally iid, i.e. \( \exists \sigma\)-algebra \( \mathcal{H} \subset \mathcal{F} \) s.t.

\[
\mathbb{P}(\tau_1 > t_1, \ldots, \tau_d > t_d | \mathcal{H}) = \prod_{k=1}^{d} \mathbb{P}(\tau_1 > t_k | \mathcal{H}), \quad \forall d \in \mathbb{N}, t_1, \ldots, t_d \in \mathbb{R}
\]

- **In a nutshell:**
  
  | conditionally iid \( \iff \) one-factor model \( \iff \) extendibility |
  |---|---|---|
  | (d) | (a), (b), (c) | (e) |

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Exchangeability criterion (necessary for CIID)

(4) All models have \(2^d - 1\) parameters (\(\lambda_I, p_I, \text{or} \tilde{p}_I\))

- **Exchangeability criterion**: The respective model is exchangeable \(\Leftrightarrow\) the parameters depend on \(I\) only through the subsets’ cardinalities \(|I|\)

- Exchangeable models have \(d\) parameters (\(\lambda_{|I|}, p_{|I|}, \text{or} \tilde{p}_{|I|}\))

- We can find a reparameterization to new parameters \((b_1, \ldots, b_d)\)

\[
\mathbb{P}(\tau_1 > t_1, \ldots, \tau_d > t_d) = \prod_{k=1}^{d} b_k^{t_{(d-k+1)} - t_{(d-k)}} \quad 0 =: t(0) \leq t(1) \leq t(2) \leq \ldots \leq t(d),
\]

where either \(t_i \in [0, \infty) [\mathcal{E}\text{-case}] \) or \(t_i \in \mathbb{N}_0 [\mathcal{G}\text{-cases}]\)
Exchangeability criterion (necessary for CIID)

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where either \(t_i \in [0, \infty)\) [\(\mathcal{E}\)-case] or \(t_i \in \mathbb{N}_0\) [\(\mathcal{G}\)-cases]

**Questions:**

- Where do the new parameters live?
- When can we extend the dimension (necessary and sufficient criterion)?
Reparameterization (e.g. narrow-sense geometric)

old parameters: $p_{1,d}, \ldots, p_{d,d}$

\[
\begin{align*}
  a_1 &:= p_{1,1} = p_{1,d}^{(d-1)} \cdots p_{d,d}^{(d-1)} \\
  a_2 &:= p_{1,2} = \prod_{i=1}^{d-1} p_{i,d}^{(d-k)} \\
  a_3 &:= p_{1,3} = \prod_{i=1}^{d-2} p_{i,d}^{(d-i-1)} \\
  a_{d-1} &:= p_{1,d-1} = p_{1,d} p_{2,d} \\
  a_d &:= p_{1,d} 
\end{align*}
\]
Reparameterization (e.g. narrow-sense geometric)

old parameters: \( p_{1,d}, \ldots, p_{d,d} \) \( \Longleftrightarrow \) new parameters: \( a_1, \ldots, a_d \)

\[
\begin{align*}
p_{1,1} & := p_{1,1} = p_{1,d} \binom{d-1}{0} \cdots p_{d,d} \binom{d-1}{d-1} \\
\vdots & \quad \vdots \\
p_{1,d-1} & \quad \cdots \quad p_{d-d-1,d-1} & a_{d-1} := p_{1,d-1} = p_{1,d} p_{2,d} \\
p_{1,d} & \quad p_{2,d} & \quad \cdots & \quad p_{d-1,d} & \quad p_{d,d} & a_d := p_{1,d}
\end{align*}
\]
Exchangeable narrow-sense geometric

**Theorem:** Representation of the exchangeable $G^N$ distribution

- Consider the exch. family of narrow-sense geometric survival functions. Then,

$$
G^N, \mathcal{X} = \{ \bar{F}_{n_1, \ldots, n_d} = \prod_{k=1}^{d} b_k^{n_{d-k+1} - n_{d-k}} | (1, b_1, \ldots, b_d) \in \mathcal{LM}_{d+1} \}
$$

- $p_{1,d}, \ldots, p_{d,d} \in (0, 1]$, with $\prod_{i=1}^{d} p_{i,d} < 1$ and $(1, b_1, \ldots, b_d) \in \mathcal{LM}_{d+1}$, are related:

$$
\frac{b_k}{b_{k-1}} = a_k = \prod_{i=1}^{d-k+1} p_{i,d}^{(d-k)_{i-1}}, \quad k = 1, \ldots, d,
$$

$$
p_{k,d} = \prod_{i=1}^{k} a_{d-i+1} (-1)^{(k-i)(k-1)_{i-1}}, \quad k = 1, \ldots, d
$$

The parameters $\{b_k\}_{k=1}^{d}$ are given by $b_k = \prod_{i=1}^{k} a_i, \ k \in \mathbb{N}$

(c) Mai, Scherer, Shenkman
Exchangeability criterion (necessary for CIID)

- Marshall–Olkin and wide-sense geometric law are reparameterized similarly
- With $0 =: t(0) \leq t(1) \leq t(2) \leq \ldots \leq t(d)$

\[
\mathbb{P}(\tau_1 > t_1, \ldots, \tau_d > t_d) = \prod_{k=1}^{d} b_k^{t(d-k+1) - t(d-k)},
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where either $t_i \in [0, \infty) \ [\mathcal{E}\text{-case}]$ or $t_i \in \mathbb{N}_0 \ [\mathcal{G}\text{-cases}]$
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\mathbb{P}(\tau_1 > t_1, \ldots, \tau_d > t_d) = \prod_{k=1}^{d} b_{t_k}^{t_{d-k+1} - t_{d-k}},
\]

where either \(t_i \in [0, \infty) \ [\mathcal{E}\text{-case}] \) or \(t_i \in \mathbb{N}_0 \ [\mathcal{G}\text{-cases}]\)

**Theorem**: This is a proper survival function in dimension \(d \iff\)

- \(\mathcal{G}^N\) and Marshall–Olkin: \((1, b_1, \ldots, b_d)\) is \(d\)-log-monotone
- \(\mathcal{G}^W:\) \((1, b_1, \ldots, b_d)\) is \(d\)-monotone

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Exchangeability criterion (necessary for CIID)

- Marshall–Olkin and wide-sense geometric law are reparameterized similarly
- With $0 =: t(0) \leq t(1) \leq t(2) \leq \ldots \leq t(d)$

\[
P(\tau_1 > t_1, \ldots, \tau_d > t_d) = \prod_{k=1}^{d} b_k^{t(d-k+1)-t(d-k)},
\]

where either $t_i \in [0, \infty)$ [E-case] or $t_i \in \mathbb{N}_0$ [G-cases]

**Theorem:** This is a proper survival function in dimension $d \iff$

- $\mathcal{G}^\mathcal{N}$ and Marshall–Olkin: $(1, b_1, \ldots, b_d)$ is $d$-log-monotone
- $\mathcal{G}^\mathcal{W}$: $(1, b_1, \ldots, b_d)$ is $d$-monotone

**Corollary:** This is an extendible survival function (i.e. for all $d \geq 2$) $\iff$

- $\mathcal{G}^\mathcal{N}$ and Marshall–Olkin: $(1, b_1, \ldots)$ is completely log-monotone
- $\mathcal{G}^\mathcal{W}$: $(1, b_1, \ldots)$ is completely monotone

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Finding a one-factor model

(4.1) Marshall–Olkin and narrow-sense geometric $\mathcal{G}^N$

**Lemma:** The following statements are equivalent

1. $\{b_k\}_{k \in \mathbb{N}_0}$ completely log-monotone with $b_0 = 1$

2. $b_k = e^{-\Psi(k)}$, $k = 0, 1, \ldots$, for a Bernstein function $\Psi$

3. $b_k = \mathbb{E}[e^{-kZ}]$, $k = 0, 1, \ldots$, for an inf. divis. r.v. $Z$ on $[0, \infty]$

4. $b_k = \mathbb{E}[e^{-kH_1}]$, $k = 0, 1, \ldots$, for a (unique) Lévy subordinator $\{H_t\}$

⇒ To each parameter sequence, associate a unique Lévy subordinator $\{H_t\}_{t \geq 0}$

⇒ This will serve as the stochastic factor in the “Lévy-frailty construction”
Lévy-frailty construction

(4.1) Marshall–Olkin and narrow-sense geometric $\mathcal{G}^N$

- Consider parameters $\{b_k\}_{k \in \mathbb{N}_0}$ completely log-monotone with $b_0 = 1$
- Let $\{H_t\}_{t \geq 0}$ be the (unique) associated Lévy subordinator with $b_k = \mathbb{E}[e^{-kH_1}]$
- Let $\{E_k\}_{k=1,\ldots,d}$ be iid $\mathcal{E}(1)$
- Define $(\tau_1, \ldots, \tau_d)$ via
  \[ \tau_k := \inf\{t > 0 : H_t > E_k\}, \quad k = 1, \ldots, d \]
Lévy-frailty construction

(4.1) **Marshall–Olkin and narrow-sense geometric $G^N$**

- Consider parameters $\{b_k\}_{k \in \mathbb{N}_0}$ completely log-monotone with $b_0 = 1$
- Let $\{H_t\}_{t \geq 0}$ be the (unique) associated Lévy subordinator with $b_k = \mathbb{E}[e^{-kH_1}]$
- Let $\{E_k\}_{k=1,...,d}$ be iid $E(1)$
- Define $(\tau_1, \ldots, \tau_d)$ via
  \[ \tau_k := \inf\{t > 0 : H_t > E_k\}, \quad k = 1, \ldots, d \]

**Theorem:**

⇒ $(\tau_1, \ldots, \tau_d)$ has the desired one-factor Marshall–Olkin law

⇒ $(\lceil \tau_1 \rceil, \ldots, \lceil \tau_d \rceil)$ has the desired one-factor $G^N$ law

\[ \mathbb{P}(\tau_1 > t_1, \ldots, \tau_d > t_d) = \prod_{k=1}^{d} b_k^{t_{(d-k+1)} - t_{(d-k)}}, \quad t_i \in [0, \infty) \text{ resp. } t_i \in \mathbb{N}_0 \]

⇒ Every one-factor Marshall–Olkin$/G^N$ law can be constructed as above

(c) Mai, Scherer, Shenkman
Lévy-frailty construction
Lévy-frailty construction
Lévy-frailty construction

\[ H_t = X_2 X_1 X_3 = X_4 \]

E_3
E_1
E_4
E_2

X_2 \quad X_1 = X_4 \quad X_3

Time t
Random walk construction

(4.II) **Wide-sense geometric** $G^W$

Consider parameters $\{b_k\}_{k=0,1,...}$ completely monotone with $b_0 = 1$

\[\Leftrightarrow \exists \! \text{ r.v. } X \text{ on } [0, 1] \text{ s.t. } b_k = \mathbb{E}[X^k] \text{ [Hausdorff (1921)]}\]

\[\Leftrightarrow \exists \! \text{ r.v. } Z \text{ on } [0, \infty] \text{ s.t. } b_k = \mathbb{E}[e^{-kZ}]\]

- Let $\{E_k\}_{k=1,...,d}$ be iid $\mathcal{E}(1)$
- Define $(\tau_1, \ldots, \tau_d)$ via

\[\tau_k := \min\{n \in \mathbb{N} : Z_1 + \ldots + Z_n > E_k\}, \quad k = 1, \ldots, d\]
Random walk construction

(4.II) **Wide-sense geometric** $G^W$

Consider parameters $\{b_k\}_{k=0,1,...}$ completely monotone with $b_0 = 1$

$\Leftrightarrow \exists ! \text{ r.v. } X \text{ on } [0, 1] \text{ s.t. } b_k = E[X^k] \text{ [Hausdorff (1921)]}$

$\Leftrightarrow \exists ! \text{ r.v. } Z \text{ on } [0, \infty] \text{ s.t. } b_k = E[e^{-kZ}]$

- Let $\{E_k\}_{k=1,...,d}$ be iid $\mathcal{E}(1)$
- Define $(\tau_1, \ldots, \tau_d)$ via
  \[ \tau_k := \min\{n \in \mathbb{N} : Z_1 + \ldots + Z_n > E_k\}, \quad k = 1, \ldots, d \]

**Theorem:**

$\Rightarrow (\tau_1, \ldots, \tau_d)$ has the desired one-factor $G^W$ law

\[ \mathbb{P}(\tau_1 > t_1, \ldots, \tau_d > t_d) = \prod_{k=1}^{d} b_k^{t_{d-k+1} - t_{d-k}}, \quad t_i \in \mathbb{N}_0 \]

$\Rightarrow$ Every one-factor $G^W$ law can be constructed as above

(c) Mai, Scherer, Shenkman
Summary
Results in a nutshell

- exponential
- exchangeable
- conditionally iid
- lack-of-memory
Summary
Results in a nutshell

exponential

lack-of-memory
(Marshall-Olkin distribution)

exchangeable
(log-monotone parameters)

conditionally iid
(Lévy subordinator)
Summary
Results in a nutshell

- **exponential** (Marshall-Olkin distribution)
- **exchangeable**
- conditionally **iid** (Lévy subordinator)
- **geometric** (narrow sense)
- **lack-of-memory** (Marshall-Olkin distribution)
- (narrow sense)
- (log-monotone parameters)
- (random walk with ID increments)

(c) Mai, Scherer, Shenkman
Summary
Results in a nutshell

- exponential (Marshall-Olkin distribution)
- geometric (narrow sense)
- geometric (wide sense)
- exchangeable (log-monotone parameters)
- conditionally iid (Lévy subordinator)
- (random walk with ID increments)
- lack-of-memory (log-monotone parameters)
- (d-monotone parameters)

(c) Mai, Scherer, Shenkman
(5) **Base case: Lévy-frailty model**

\[ \tau_k := \inf \{ t \geq 0 : H_{-\frac{\log(1-p(t))}{\Psi(1)}} \geq E_k \}, \quad E_k \overset{iid}{\sim} \mathcal{E}(1), \quad k \in \mathbb{N} \]

- \( H = \{H_t\}_{t \geq 0} \) a Lévy subordinator with Laplace exponent \( \Psi(x) \)
- Term structure of default probabilities \( t \mapsto p(t) := \mathbb{P}(\tau_k \leq t) \)
Portfolio default models

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- Term structure of default probabilities \( t \mapsto p(t) := \mathbb{P}(\tau_k \leq t) \)

**Properties:**

- Survival copula: \((u_1, \ldots, u_d) \mapsto \prod_{i=1}^{d} u_{(i)}^{(\Psi(i) - \Psi(i-1))/\Psi(1)}\)
- \( t \mapsto p(t) := \mathbb{P}(\tau_k \leq t) \) is marginal law
- Singular component: \( \mathbb{P}(\tau_1 = \ldots = \tau_k) = \frac{1}{\Psi(k)} \sum_{i=0}^{k} \binom{k}{i} (-1)^{i+1} \Psi(i) > 0, \quad k \geq 2 \)
- Lower-tail dependence of \((\tau_i, \tau_j)\): \( \lambda_l = 2 - \Psi(2)/\Psi(1) \)
- Dynamic model, but time-homogeneous innovations

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LHP approximation

**Lemma**: Distribution of $L_t = \frac{1}{d} \sum_{k=1}^{d} 1\{\tau_k \leq t\}$

- Define $F_t = 1 - \exp(-H - \log(1-p(t))/\Psi(1))$
- Exact distribution of $L_t$:

$$
\mathbb{P}\left(L_t = \frac{k}{d}\right) = \binom{d}{k} \mathbb{E}\left[F_t^k (1 - F_t)^{d-k}\right], \quad k = 0, 1, \ldots, d
$$
LHP approximation

**Lemma:** Distribution of $L_t = \frac{1}{d} \sum_{k=1}^{d} 1\{\tau_k \leq t\}$

- Define $F_t = 1 - \exp(-H \log(1-p(t))/\Psi(1))$
- Exact distribution of $L_t$:
  \[ P\left(L_t = \frac{k}{d}\right) = \binom{d}{k} \mathbb{E}\left[ F_t^k (1 - F_t)^{d-k} \right], \quad k = 0, 1, \ldots, d \]
- Glivenko–Cantelli implies:
  \[ P\left( \lim_{d \to \infty} \sup_{t \geq 0} |F_t - L_t| = 0 \right) = 1 \]
- For $d \gg 2$, this justifies:
  \[ \mathbb{E}[f(L_t)] = \int_{[0,1]} f(x) P(L_t \in dx) \approx \int_{[0,1]} f(x) P(F_t \in dx) \]
  \[ \Rightarrow \text{Distribution of } F_t \text{ is sufficient to price portfolio derivatives} \]
  \[ \Rightarrow \text{CDO tranches can even be priced via FFT} \]
Beyond LOM

(5.1) Scale mixtures of Marshall–Olkin

\[ \tau_k := \inf \{ t \geq 0 : H_{M_t} \geq E_k \}, \quad E_k \overset{iid}{\sim} \mathcal{E}(1), \ k \in \mathbb{N} \]

- \( H = \{H_t\}_{t \geq 0} \) a Lévy subordinator with Laplace exponent \( \Psi(x) \)
- \( M \) a positive r.v. with Laplace transform \( \varphi \)
Beyond LOM

(5.1) Scale mixtures of Marshall–Olkin

\[ \tau_k := \inf\{t \geq 0 : H_{Mt} \geq E_k\}, \quad E_k \overset{iid}{\sim} \mathcal{E}(1), \quad k \in \mathbb{N} \]

- \( H = \{H_t\}_{t \geq 0} \) a Lévy subordinator with Laplace exponent \( \Psi(x) \)
- \( M \) a positive r.v. with Laplace transform \( \varphi \)

Properties:

- Survival copula: \((u_1, \ldots, u_d) \mapsto \varphi\left( \frac{1}{\Psi(1)} \sum_{i=1}^{d} \varphi^{-1}(u_{(i)}) \left( \Psi(i) - \Psi(i - 1) \right) \right)\)

  \( \Rightarrow \) Combination of Archimedean and Marshall-Olkin copula

- Singular component: \( \mathbb{P}(\tau_1 = \ldots = \tau_k) = \frac{1}{\Psi(k)} \sum_{i=0}^{k} \binom{k}{i} (-1)^{i+1} \Psi(i) > 0, \quad k \geq 2 \)

- Lower-tail dependence of \((\tau_i, \tau_j)\): \( \lambda_l = 2 - \frac{\Psi(2)}{\Psi(1)} \lim_{t \downarrow 0} \frac{\varphi'(t \Psi(2)/\Psi(1))}{\varphi'(t)} \)

- Dynamic model with time-inhomogeneous innovations \( \Rightarrow \text{contagion effect} \)

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Beyond LOM
(5.II) Intensity combined with jumps

\[ \tau_k := \inf\{t \geq 0 : H \int_0^t \lambda_s ds \geq E_k\}, \quad E_k \overset{iid}{\sim} \mathcal{E}(1), \quad k \in \mathbb{N} \]

- \( H = \{H_t\}_{t \geq 0} \) a Lévy subordinator with Laplace exponent \( \Psi(x) \)
- \( \{\lambda_t\}_{t \geq 0} \) a classical intensity model, e.g. basic affine process
Beyond LOM

(5.II) Intensity combined with jumps

\[ \tau_k := \inf\{ t \geq 0 : H \int_0^t \lambda_s ds \geq E_k \}, \quad E_k \overset{iid}{\sim} \mathcal{E}(1), \ k \in \mathbb{N} \]

- \( H = \{ H_t \}_{t \geq 0} \) a Lévy subordinator with Laplace exponent \( \Psi(x) \)
- \( \{ \lambda_t \}_{t \geq 0} \) a classical intensity model, e.g. basic affine process

Properties:

- Survival copula: Known, but tedious formula
- Singular component: Identical to Lévy-frailty model
- Lower-tail dependence of \((\tau_i, \tau_j)\): Positive
- **Dynamic, time-inhomogeneous innovations**
- Intensity process \( \Rightarrow \text{spread risk} \)
- Lévy subordinator \( \Rightarrow \text{jump risk} \), i.e. cataclysmic events
Beyond LOM

(5.III) Shot-noise frailty model

\[ \tau_k := \inf \{ t \geq 0 : S_t \geq E_k \}, \quad E_k \overset{iid}{\sim} \mathcal{E}(1), \quad k \in \mathbb{N} \]

- \( S \) a pathwise non-decreasing shot-noise process

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Beyond LOM
(5.III) Shot-noise frailty model

\[ \tau_k := \inf \{ t \geq 0 : S_t \geq E_k \}, \quad E_k \sim \mathcal{E}(1), \quad k \in \mathbb{N} \]

- \( S \) a pathwise non-decreasing shot-noise process

**Properties:**
- Joint survival function \( \mathbb{P}(\tau_1 > t_1, \ldots, \tau_d > t_d) \) known explicitly
- Singular component: Yes, if desired
- Lower-tail dependence of \((\tau_i, \tau_j)\): Positive, known in closed form
- Dynamic model with time-inhomogeneous innovations \( \Rightarrow \) contagion effect
Beyond LOM
(5.III) Shot-noise frailty model

- At some jump (catastrophe): Multiple defaults
- After some jump: More default activity

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Conclusion

- The conditionally iid subfamilies of the Marshall–Olkin and the multivariate geometric law have been characterized analytically via (log-)completely monotone sequences
- Solutions to moment problems provide a bridge to infinitely divisible distributions
- New stochastic models for these subfamilies, based on certain frailty models, have been constructed
- Portfolio default (and insurance) models with interesting stylized properties can be constructed based on these subfamilies and their extensions

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References

- **$G$-case**: [Mai, Scherer, Shenkman (2012)]: “Multivariate geometric distributions, (logarithmically) monotone sequences, and infinitely divisible laws”, working paper, resubmitted to JMVA

- **$E$-case**: Chapter 3 of [Mai, Scherer (2012)]: “Simulating Copulas”
Generalization to inhomogeneous univariate marginals

- **Generalization**: inhomogeneous marginal laws \( t \mapsto p_k(t) = \mathbb{P}(\tau_k \leq t) \)

- **Model**: For some market factor \( M = \{M_t\}_{t \geq 0} \), let
  \[
  \tau_k := \inf \{ t \geq 0 : U_k \leq F_t^{(k)} := \text{function}(p_k(t), M) \}\]

- Then, the **joint distribution** is
  \[
  \mathbb{P}(\tau_1 \leq t_1, \ldots, \tau_d \leq t_d) = \mathbb{E}[\mathbb{E}\left[ \prod_{k=1}^d \mathbf{1}_{\{\tau_k \leq t_k\}} \mid M \right]] = \mathbb{E}\left[ \prod_{k=1}^d F_{t_k}^{(k)} \right]
  \]
  and the copula remains the same as in the homogeneous case

- **Loss distribution via recursion**: use the classical recursion formula
  \[
  \Pi_{k}^{M,n+1}(t) := (1 - F_{t}^{(n+1)}) \Pi_{k}^{M,n}(t) + F_{t}^{(n+1)} \Pi_{k-1}^{M,n}, \quad 0 \leq n \leq d - 1,
  \]
  \[
  \Pi_{0}^{M,0} := 1, \quad \Pi_{d-1}^{M,d} = 0
  \]
  > At the end of the iteration, \( \Pi_{k}^{M,d}(t) \) is the conditional probability (given \( M \)) of having precisely \( k \) defaults until time \( t \), where \( 0 \leq k \leq d \)
  > The unconditional probability is obtained by integrating out \( M \)
Generalization to hierarchical dependence structures

• **Observation**: firms in the same category, such as ...
  - ... the same industry sector
  - ... the same geographic region
  - ... some other economic criterion

are stronger associated as firms from different categories

• **Example**: the iTraxx CDO portfolio: $d = 125$ firms from $J = 6$ industry sectors

• **Model generalization**: group specific frailty distributions $F_t^{(j)}, j = 1, \ldots, J$
Generalization to hierarchical dependence structures

- **Example (Hierarchical Archimedean copulas):** Let
  
  
  \[ M \text{ be a positive rv with Laplace transform } \psi_0(x) = \mathbb{E}[\exp(-xM)] \]
  
  \[ \Lambda^{(j)} \text{ be ind. Lévy subordinators with Laplace exponent } \Psi_j, \ j = 1, \ldots, J \]
  
  Then
  
  \[ F_t^{(j)} = 1 - \exp \left( -\Lambda_{M}^{(j)} \Psi_j^{-1} (\varphi^{-1}(1 - p(t))) \right), \quad \tau_{ji} := \inf \left\{ t \geq 0 : U_{ji} \leq F_t^{(j)} \right\} \]
  
  implies the hierarchical Archimedean (survival) copula
  
  \[ C_{\psi_0} \left( C_{\psi_1}(u_{11}, \ldots, u_{1d_1}), \ldots, C_{\psi_J}(u_{J1}, \ldots, u_{Jd_J}) \right) \]
  
  with \( \psi_j = \psi_0 \circ \Psi_j \) and \( C_{\psi}(u_1, \ldots, u_n) = \psi \left( \psi^{-1}(u_1) + \ldots + \psi^{-1}(u_n) \right) \)
Generalization to hierarchical dependence structures

- **Example (Hierarchical Lévy-frailty copulas):**

  Let $\Lambda^{(0)}, \Lambda^{(1)}, \ldots, \Lambda^{(J)}$ be ind. Lévy subordinators with Laplace exponent $\Psi_j$. Then for $a \in (0, 1)$

  $$F_t^{(j)} = 1 - \exp \left( - \Lambda^{(0)}_{a t} - \Lambda^{(j)}_{(1-a) t} \right), \quad \tau_{ji} := \inf \{ t \geq 0 : U_{ji} \leq F_t^{(j)} \}$$

  implies the (survival) copula

  $$C(\vec{u}) = C_{\Psi^0}(\vec{u})^a \left( \prod_{j=1}^J C_{\Psi_j}(u_{j1}, \ldots, u_{jn_j}) \right)^{1-a},$$

  where $\vec{u} = (u_{11}, \ldots, u_{1d_1}, \ldots, u_{J1}, \ldots, u_{Jd_J}) \in [0, 1]^{d_1+\ldots+d_J}$ and

  $$C_{\Psi}(u_1, \ldots, u_n) := \prod_{i=1}^n u_{(i)}^{\left( \Psi(i) - \Psi(i-1) \right)/\Psi(1)}$$

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