Simulation of BSDEs and Wiener Chaos Expansions

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Introduction
A BSDE is an equation of the type

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T \]

The first result was proved by É. Pardoux and S. Peng

**Theorem (Pardoux-Peng)**

*If the generator $f$ is Lipschitz, and

\[ \mathbb{E} \left[ |\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds \right] < +\infty \]

*then the BSDE has a unique square integrable solution*
Euler’s scheme for BSDEs

- For a SDE
  \[ X_{t+h} = X_t + h b(X_t) + \sigma(X_t) (B_{t+h} - B_t) \]

- For a BSDE
  \[ Y_{t-h} = Y_t + h f(Y_{t-h}, Z_{t-h}) - Z_{t-h} (B_t - B_{t-h}) \]

★ The correct equation is
  \[ Y_{t-h} = Y_t + h f(Y_{t-h}, Z_{t-h}) - Z_{t-h} (B_t - B_{t-h}) + (N_t - N_{t-h}) \]

★ The solution is given by
  \[
  Z_{t-h} = h^{-1} \mathbb{E} \left( Y_t (B_t - B_{t-h}) \mid \mathcal{F}_{t-h} \right), \\
  Y_{t-h} = \mathbb{E} \left( Y_t \mid \mathcal{F}_{t-h} \right) + h f(Y_{t-h}, Z_{t-h})
  \]
Discretization of BSDEs

- Not so easy as for SDEs
- D. Chevance 1997, V. Bally 1997
- Dynamic Programming Equation
  - A "stepwise constant" Brownian motion in this case (Ph. B., B. Delyon, J. Mémin 01 & 02)
  - The speed of convergence was proved by J. Zhang 04 and B. Bouchard and N. Touzi 04 for Markovian BSDEs
  - Error expansion by E. Gobet and C. Labart 07
  - $L^2$regularity E. Gobet, A. Makhlouf 10, Quadratic case A. Richou 11
Discretization of BSDEs

- One of the main difficulties is to compute conditional expectations
- Several approaches:
  - using Malliavin calculus, B. Bouchard and N. Touzi
  - using regression methods, E. Gobet, J.-P. Lemor and X. Warin
  - using quantization, V. Bally and G. Pagès
  - using the curbature method T. Lyons, D. Crisan and K. Manolarakis
- We propose here to use Wiener chaos expansions
  - In the spirit, it is not so far from the regression techniques
  - C. Bender and R. Denk 07
How to solve a BSDE?

- Two steps:
  - The generator does not depend on $(Y, Z)$
  - Fixed point argument — Picard iterations

- $\xi$, $\{f(s)\}_{0 \leq s \leq T}$ square integrable

$$Y_t = \xi + \int_t^T f(s) \, ds - \int_t^T Z_s \, dB_s, \quad 0 \leq t \leq T.$$ 

- The solution is given by

$$Y_t = \mathbb{E} \left( \xi + \int_t^T f(s) \, ds \bigg| \mathcal{F}_t \right).$$
How to solve a BSDE?

- Setting $F = \xi + \int_0^T f(s) \, ds$, we have

$$Y_t = \mathbb{E}(F \mid \mathcal{F}_t) - \int_0^t f(s) \, ds$$

$$F = \mathbb{E}[F] + \int_0^T Z_s \, dB_s.$$

- Finally, the solution is given by

$$Y_t = \mathbb{E}(F \mid \mathcal{F}_t) - \int_0^t f(s) \, ds$$

$$Z_t = \mathbb{E}(D_tF \mid \mathcal{F}_t) = D_t Y_t$$

- Explicit formulae using Wiener chaos expansions + Picard iterations
Simulation of BSDEs and Wiener Chaos Expansions

The Algorithm
Multiple Wiener Integrals

- \((\Omega, \mathcal{F}, \mathbb{P})\) complete probability space
- \(B\) real valued Brownian motion
- \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) augmented filtration of \(B : \mathcal{F} = \mathcal{F}_T\)

If \(F \in L^2(\Omega, \mathcal{F}, \mathbb{P}),\)

\[
F = \mathbb{E}[F] + \int_0^T u_1(s_1) dB_{s_1} + \int_0^T \int_0^{s_2} u_2(s_2, s_1) dB_{s_1} dB_{s_2} \\
+ \ldots + \int_0^T \int_0^{s_n} \cdots \int_0^{s_2} u_n(s_n, \ldots, s_1) dB_{s_1} \ldots dB_{s_n} + \ldots
\]

\(\star\) \(u_n\) is a deterministic function defined on

\[
\mathcal{S}_n(T) := \{(s_1, \ldots, s_n) \in [0, T]^n : 0 < s_1 < \ldots < s_n < T\}
\]
Multiple Wiener Integrals

- If \( u_n \in L^2(S_n(T)) \), we set
  \[
  J_n(u_n) := \int_0^T \int_0^{s_n} \cdots \int_0^{s_2} u_n(s_n, \ldots, s_1) dB_{s_1} \cdots dB_{s_n},
  \]
  \( \mathcal{H}_n := \text{vect} (J_n(u_n) : u_n \in L^2(S_n(T))) \)

- We have
  \[
  \mathbb{E} [J_n(u_n)^2] = \int_0^{s_n} \cdots \int_0^{s_2} u_n^2(s_n, \ldots, s_1) ds_1 \cdots ds_n
  \]
  \[
  \mathbb{E} [J_n(u_n)J_m(u_m)] = \delta_{nm} \int_0^{s_n} \cdots \int_0^{s_2} u_n(s_n, \ldots, s_1)u_m(s_n, \ldots, s_1) ds_1 \cdots ds_n
  \]
Multiple Wiener Integrals

- To summarize, with $\mathcal{H}_0 = \mathbb{R}$,

$$L^2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{n \geq 0} \mathcal{H}_n$$

Notations: for $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$

- $P_n(F) = J_n(u_n)$ is the projection of $F$ on $\mathcal{H}_n$,
- $C_p(F)$ denotes the chaos decomposition of $F$ up to order $p$

$$C_p(F) = \mathbb{E}[F] + \sum_{k=1}^{p} P_k(F) = \mathbb{E}[F] + \sum_{k=1}^{p} J_k(u_k)$$

$$= \mathbb{E}[F] + \int_0^T u_1(s_1) dB_{s_1} + \ldots + \int_0^T \cdots \int_0^{s_2} u_p(s_p, \ldots, s_1) dB_{s_1} \ldots dB_{s_p}$$

$$\mathbb{E}[F^2] = \mathbb{E}[F]^2 + \sum_{n \geq 1} \mathbb{E}[P_n(F)^2]$$
A simple BSDE

- Let us consider the BSDE

\[ Y_t = \xi + \int_t^T f(s) \, ds - \int_t^T Z_s \, dB_s, \quad 0 \leq t \leq T. \]

- whose solution is given by

\[ Y_t = \mathbb{E}(F | \mathcal{F}_t) - \int_0^t f(s) \, ds, \]
\[ Z_t = \mathbb{E}(D_t F | \mathcal{F}_t) = D_t Y_t, \]

where

\[ F = \xi + \int_0^T f(s) \, ds. \]
Ideas of the algorithm

- The first idea consists in replacing

\[ F = \mathbb{E}[F] + \sum_{k \geq 1} P_k(F) = \mathbb{E}[F] + \sum_{k \geq 1} J_k(u_k), \]

by the random variable

\[ C_p(F) = \mathbb{E}[F] + \sum_{k=1}^{p} P_k(F) = \mathbb{E}[F] + \sum_{k=1}^{p} J_k(u_k) \]

- Namely,

\[ Y_t \sim \mathbb{E}_t (C_p(F)) = \mathbb{E}[F] + \sum_{k=1}^{p} \mathbb{E}_t (J_k(u_k)) \]
Ideas of the algorithm

- The second idea is to replace the functions $u_k$ by piecewise constant functions:

$$u_1(s) \approx \sum_{i=1}^{n} \lambda_i 1_{t_{i-1}, t_i}(s),$$

$$u_2(s, t) \approx \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_{i,j} 1_{t_{i-1}, t_i}(s) 1_{t_{j-1}, t_j}(t).$$

- With this approximation,

$$J_k(u_k) = \int_0^T \cdots \int_0^{s_2} u_k(s_k, \ldots, s_1) dB_{s_1} \cdots dB_{s_k}$$

can be expressed as a combination of Hermite polynomials.
Hermite polynomials

- The Hermite polynomials are defined by the expansion
  
  \[ e^{xt - t^2/2} = \sum_{n\geq 0} H_n(x) \frac{t^n}{n!} = \sum_{n\geq 0} K_n(x) t^n \]

- \( H_2(x) = x^2 - 1, \ K_1(x) = \frac{1}{2} (x^2 - 1). \)

- \( \left( \sqrt{n!} \ K_n \right)_{n \geq 0} \) is an orthonormal basis of \( L^2(\mathbb{R}, \mu) \)
  - \( \mu \) normal distribution \( \mathcal{N}(0, 1) \)

- Let \( (g_n)_{n \geq 1} \) be an orthonormal basis of \( L^2(0, T) \)
- \( \mathcal{H}_n \) is the closure of the vector field spanned by
  
  \[ \left\{ \prod_{i \geq 1} \sqrt{n_i!} \ K_{n_i} \left( \int_0^T g_i(s) dB_s \right) : \|(n_i)_{i \geq 1}\| := \sum_{i \geq 1} n_i = n \right\} \]
Hermite polynomials

- Let $\Lambda$ be the set of all sequences of integers $(n_i)_{i \geq 1}$ such that
  $$|n| = \sum_{i \geq 1} n_i < +\infty$$

- The set
  $$\left\{ \prod_{i \geq 1} \sqrt{n_i !} K_{n_i} \left( \int_0^T g_i(s) dB_s \right) : n = (n_i)_{i \geq 1} \in \Lambda \right\}$$
  is an orthonormal basis of $L^2(\Omega, \mathcal{F}, \mathbb{P})$

- For $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$,
  $$F = \mathbb{E}[F] + \sum_{k \geq 1} \sum_{|n| = k} d^n_k \prod_{i \geq 1} K_{n_i} \left( \int_0^T g_i(s) dB_s \right)$$
  $$d^n_k = n! \mathbb{E} \left[ F \times \prod_{i \geq 1} K_{n_i} \left( \int_0^T g_i(s) dB_s \right) \right], \quad n! = \prod_{i \geq 1} n_i !$$
A simple BSDE

- We consider the BSDE

\[ Y_t = \xi + \int_t^T f(s) \, ds - \int_t^T Z_s \, dB_s, \quad 0 \leq t \leq T \]

- The solution is given by

\[ Y_t = \mathbb{E}(F | \mathcal{F}_t) - \int_0^t f(s) \, ds, \quad Z_t = D_t \mathbb{E}(F | \mathcal{F}_t), \quad F = \xi + \int_0^T f(s) \, ds \]

- Compute \( Y \) and \( Z \) with the decomposition

\[
F = \mathbb{E}[F] + \sum_{k \geq 1} \sum_{|n|=k} d^n_k \prod_{i \geq 1} K_{n_i} \left( \int_0^T g_i(s) dB_s \right) \\
d^n_k = n! \mathbb{E} \left[ F \prod_{i \geq 1} K_{n_i} \left( \int_0^T g_i(s) dB_s \right) \right], \quad n! = \prod_{i \geq 1} n_i!
\]
Choice of the basis

- We choose an orthonormal basis of $L^2(0, T)$ for which computations are easy.
- The first $N$ functions, $g_1, \ldots, g_N$, are given by
  \[ g_i(t) = (t_i - t_{i-1})^{-1/2} \mathbf{1}_{[t_{i-1}, t_i]}(t), \quad t_i = i \frac{T}{N} = ih, \quad i = 1, \ldots, n \]
- We complete these functions into an orthonormal basis of $L^2(0, T)$, $(g_i)_{i \geq 1}$
  - We can consider the Haar basis on each interval $(t_{i-1}, t_i)$
Approximation of $F$

- We consider a finite numbers of chaos $p$

$$F \simeq C_p(F) = \mathbb{E}[F] + \sum_{k=1}^{p} \sum_{|n|=k} d^n_k \prod_{i \geq 1} K_{n_i} \left( \int_0^T g_i(s) dB_s \right)$$

- We work only with the functions $g_1, \ldots, g_N$

$$F \simeq C_p(F) \simeq C^N_p(F) = \mathbb{E}[F] + \sum_{k=1}^{p} \sum_{|n|=k} d^n_k \prod_{i=1}^{n} K_{n_i} \left( \int_0^T g_i(s) dB_s \right)$$

$$= \mathbb{E}[F] + \sum_{k=1}^{p} \sum_{|n|=k} d^n_k \prod_{i=1}^{n} K_{n_i} (G_i)$$

$$d^n_k = n_1! \ldots n_N! \times \mathbb{E}[F K_{n_1}(G_1) \ldots K_{n_N}(G_N)]$$

with the notations

$$n = (n_1, \ldots, n_N), \quad G_i = (t_i - t_{i-1})^{-1/2} (B_{t_i} - B_{t_{i-1}})$$
Approximation of $F$

- Roughly speaking, we consider that $\mathcal{H}_k$ is generated by
  $$\{K_{n_1}(G_1)\ldots K_{n_N}(G_N) : n_1 + \ldots + n_N = k\}$$

Exemple

- For $p = 2$,
  $$C^N_2(F) = \mathbb{E}[F] + \sum_{j=1}^{N} d_{1}^{e_j} K_1(G_j) + \sum_{j=1}^{N} \sum_{i=1}^{j-1} d_{2}^{e_{ij}} K_1(G_i) K_1(G_j) + \sum_{j=1}^{N} d_{2}^{2e_j} K_2(G_j),$$

  where $(e_j)_{1 \leq j \leq N}$ is the canonical basis of $\mathbb{R}^N$, and $e_{ij} = e_i + e_j$.

- For $j = 1, \ldots, N$ and $i = 1, \ldots, j - 1$
  $$d_{1}^{e_j} = \mathbb{E}(FK_1(G_j)), \quad d_{2}^{e_{ij}} = \mathbb{E}(FK_1(G_i)K_1(G_j)), \quad d_{2}^{2e_j} = 2\mathbb{E}(FK_2(G_j)).$$
Approximation of \((Y, Z)\): continuous time

- For \(1 \leq r \leq N\) and \(t_{r-1} < t \leq t_r\), with \(n(r) = (n_1, \ldots, n_r)\), we have

\[
\mathbb{E}_t \left( C^N_p (F) \right) = \mathbb{E} [F] + \sum_{k=1}^{p} \sum_{|n(r)|=k} d_k^n \prod_{i<r} K_{n_i} (G_i) \times \left( \frac{t - t_{r-1}}{h} \right)^{\frac{n_r}{2}} K_{n_r} \left( \frac{B_t - B_{t_{r-1}}}{\sqrt{t - t_{r-1}}} \right),
\]

and we set

\[
Y^{p,N}_t = \mathbb{E}_t \left( C^N_p (F) \right) - h \sum_{i=1}^{r-1} f(t_i) - (t - t_{r-1}) f(t_{r-1}), \quad Y^{p,N}_0 = \mathbb{E} (F),
\]

since for \(0 \leq t \leq T\),

\[
Y_t = \mathbb{E}_t (F) - \int_0^t f(s) \, ds.
\]
Approximation of \((Y, Z)\): continuous time

- For \(1 \leq r \leq N\) and \(t_{r-1} < t < t_r\), we set

\[
Z_t^{p,N} = E_t \left( D_t C_{p}^{N}(F) \right) = h^{-1/2} \sum_{k=1}^{p} \sum_{\substack{|n(r)|=k \, n_r>0}} d_k^n \prod_{i<r} K_{n_i}(G_i) \times \left( \frac{t - t_{r-1}}{h} \right)^{\frac{n_r-1}{2}} K_{n_r-1} \left( \frac{B_t - B_{t_{r-1}}}{\sqrt{t - t_{r-1}}} \right),
\]

since \(Z_t = E_t (D_tF)\).

- For \(t = t_r\), \(r = 0, \ldots, N\), it depends on the choice

\[
D_t(B_v - B_u) = 1_{[u,v]}(t) \quad \text{or} \quad D_t(B_v - B_u) = 1_{[u,v]}(t).
\]

- We choose the first one. To define \(Z_0^{p,N}\) we set

\[
Z_0^{p,N} = \lim_{t \to 0^+} Z_t^{p,N} = \frac{d_1^{e_1}}{\sqrt{h}} = \frac{E[F B_h]}{h}.
\]
Approximation of \((Y, Z)\): continuous time

- For \(r = 1, \ldots, N\),

\[
Y_{t_r}^{p,N} = \mathbb{E}[F] + \sum_{k=1}^{p} \sum_{|n(r)|=k} d_k^n \prod_{i \leq r} K_{n_i}(G_i) - h \sum_{i=1}^{r} f(t_i),
\]

\[
Y_0^{p,N} = \mathbb{E}[F] = d_0,
\]

\[
Z_{t_r}^{p,N} = h^{-1/2} \sum_{k=1}^{p} \sum_{|n(r)|=k} d_k^n \prod_{i < r} K_{n_i}(G_i) \times K_{n_{r-1}}(G_r),
\]

\[
Z_0^{p,N} = \frac{d_1^{e_1}}{\sqrt{h}} = \frac{\mathbb{E}[F B_h]}{h}.
\]
Approximation of \((Y, Z)\): continuous time

- \(t \mapsto Y_{t}^{p,N}\) is continuous on \([0, T]\)
  \[
  Y_{0}^{p,N} = \mathbb{E}[F] = d_0
  \]
- \(t \mapsto Y_{t}^{p,N}\) is left-continuous on \([0, T]\) and continuous at \(t = 0\)
  \[
  Z_{0}^{p,N} = \lim_{t \to 0} Z_{t}^{p,N} = \frac{d_{1}^{e_1}}{\sqrt{h}} = \frac{\mathbb{E}[F B_h]}{h}.
  \]
- Case \(p = 2\), for \(1 \leq r \leq N\),
  \[
  Y_{t,r}^{2,N} = \mathbb{E}[F] + \sum_{j=1}^{r} d_{1}^{e_j} K_1(G_j) + \sum_{j=1}^{r} \sum_{i=1}^{j-1} d_{2}^{e_{ij}} K_1(G_i) K_1(G_j) + \sum_{j=1}^{r} d_{2}^{e_j} K_2(G_j),
  \]
  \[
  Z_{t,r}^{2,N} = h^{-1/2} \left( d_{1}^{e_r} + d_{2}^{e_r} K_1(G_r) + \sum_{i=1}^{r-1} d_{2}^{e_{ir}} K_1(G_i) \right).
  \]
Formulas when $d > 1$

\[
\mathbb{E}_{t_r}(C^N_p(F)) = \mathbb{E}[F] + \sum_{k=1}^{p} \sum_{|n(r)|=k} d^n_k \prod_{i \leq r} \prod_{1 \leq j \leq d} K_{n'_i} \left( G^i_j \right),
\]

\[
Y^p,N_{t_r} = \mathbb{E}_{t_r}(C^N_p(F)) - h \sum_{i=1}^{r} f(t_i),
\]

\[
Z^p,N_{t_r} = h^{-1/2} \sum_{k=1}^{p} \sum_{|n(r)|=k} d^n_k \prod_{i < r} \prod_{1 \leq j \leq d} K_{n'_i} \left( G^i_j \right) \times K_{n'_r-1} \left( G^i_r \right) \prod_{j \neq l} K_{n'_r} \left( G^i_r \right).
\]

The Algorithm

- Starting from $M$ trajectories of the BM $B$, we get $M$ trajectories of $F$

- We compute

$$
\hat{d}_0 = \hat{E}[F] = \frac{1}{M} \sum_{m=1}^{M} F^m
$$

$$
\hat{d}_n^k = n! \hat{E}[F K_{n_1}(G_1) \ldots K_{n_N}(G_N)] = n! \times \frac{1}{M} \sum_{m=1}^{M} F^m K_{n_1}(G_1^m) \ldots K_{n_N}(G_N^m)
$$

- We get $M$ trajectories of $(Y, Z)$

$$
Y_{tr}^{p,N,m} = \hat{d}_0 + \sum_{k=1}^{p} \sum_{|n(r)|=k} \hat{d}_n^k \prod_{i \leq r} K_{n_i}(G_i^m) - h \sum_{i=1}^{r} f^m(t_{i-1}),
$$

$$
Z_{tr}^{p,N,m} = h^{-1/2} \sum_{k=1}^{p} \sum_{|n(r)|=k} \hat{d}_n^k \prod_{i < r} K_{n_i}(G_i^m) \times K_{n_r-1}(G_r^m).
$$
Forward vs Backward

• We choose to write the solution in a forward way:

\[ Y_t = \mathbb{E}_t(F) - \int_0^t f(s) \, ds, \quad F = \xi + \int_0^T f(s) \, ds \]

instead of using the classical backward formula

\[ Y_t = \mathbb{E}_t \left( \xi + \int_t^T f(s) \, ds \right) \]

With the forward formula

• We have only to compute the Wiener chaos expansion of a single random variable

• But we break down the backward structure of the equation
A BSDE with a generator

- We consider now the BSDE

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T. \]

- We assume that \( f : [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \) is Lipschitz

\[ |f(t, y, z) - f(t, y', z')| \leq L_f (|y - y'| + |z - z'|). \]

- The algorithm consists in two steps
  - The Picard scheme to remove the generator
  - The previous approximation
Picard Iterations

- \((Y^0, Z^0) = (0, 0)\) and for \(q \geq 0\)

\[
Y_t^{q+1} = \xi + \int_t^T f(s, Y^q_s, Z^q_s) \, ds - \int_t^T Z^{q+1}_s \cdot dB_s, \quad 0 \leq t \leq T.
\]

- Let \((Y^{q,p,N}, Z^{q,p,N})\) be the approximation of \((Y^q, Z^q)\)
  - \((Y^{0,p,N}, Z^{0,p,N}) = (0, 0)\)
  - \(F^{q,p,N} = \xi + \int_0^T f(s, Y^{q,p,N}_s, Z^{q,p,N}_s) \, ds\)
Previous Approximation

- We construct, with the explicit formulas, for \( r = 0, \ldots, N \)

\[
Y_{tr}^{q+1,p,N} = \mathbb{E} \left( C_p^N (F_{q,p,N}) \mid \mathcal{F}_{tr} \right) - h \sum_{i=1}^{r} f \left( t_{i-1}, Y_{t_{i-1}}^{q,p,N}, Z_{t_{i-1}}^{q,p,N} \right)
\]

\[
Z_{tr}^{q+1,p,N} = D_{tr} Y_{tr}^{q+1,p,N} = D_{tr} \mathbb{E} \left( C_p^N (F_{q,p,N}) \mid \mathcal{F}_{tr} \right)
\]

- Starting with \( M \) trajectories of \( B \)
  - the coefficients of the Wiener expansions are computed by Monte-Carlo
  - we obtain \( M \) trajectories of the process \((Y_{q,p,N,M}, Z_{q,p,N,M})\)
Simulation of BSDEs and Wiener Chaos Expansions

Numerical Examples
Linear Driver - Financial Benchmark

• Black-Scholes model in dimension $d = 1$

\[ S_t = S_0 e^{(r - \frac{1}{2} \sigma^2) t + \sigma W_t}, \quad \forall t \in [0, T]. \]

• Linear generator $f(t, y, z) = -ry$

• We consider a Discrete Down and Out Barrier Call option

\[ \xi = (S_T - K)^+ 1_{\forall n \in [0, N] S_{t_n} \geq L} \]

• The parameters are

\[ r = 0.01, \quad \sigma = 0.2, \quad T = 1, \quad K = 0.9, \quad L = 0.85, \quad S_0 = 1, \quad N = 20 \]

• By a variance reduction Monte Carlo method

\[ Y_0 = 0.134267, \quad \delta_0 = \frac{Z_0}{\sigma S_0} = 0.8327 \]
Figure: Evolution of $Y_{0}^{q,p,N,M}$ and $\delta_{0} := \frac{Z_{0}^{q,p,N,M}}{\sigma S_{0}}$ w.r.t. $\log(M)$ when $N = 20$, $p = 2$, $q = 5$ - Discrete Down and Out Barrier Call option
Non linear driver - Financial Benchmark

- Black-Scholes model in dimension $d = 5$

\[ \forall i = 1, \ldots, 5 \quad S^i_t = S^i_0 e^{(\mu^i - (\sigma^i)^2/2)t + \sigma^i B^i_t}. \]

- The correlation matrix of $B$ is given by

\[ \langle B^i, B^j \rangle_t = c_{i,j}t = \rho t 1_{i \neq j} + t 1_{i = j}, \quad -\frac{1}{4} < \rho < 1 \]

- The borrowing rate $R$ is higher than the bond one $r$

- The driver is given by

\[ f(t, y, z) = -ry - \theta \cdot z + (R - r)(y - \sum_{i=1}^{5}(\Sigma^{-1} z)_i)^- \]

where $\theta = \theta := \Sigma^{-1}(\mu - r 1), \Sigma_{ij} = \sigma^i L_{ij}, C = LL^*$

- We consider a Put Basket Option: $\xi = (K - \frac{1}{5} \sum_{i=1}^{5} S^i_T)^+$
Non linear driver - Financial Benchmark

- Comparison with results from É. Gobet & C. Labart 2010
- CPU time = 1660 when $M = 500000$ and $N = 20$

Figure: Evolution of $Y_0^{q,p,N,M}$ and $\delta_0(1)$ w.r.t. $\log(M)$ when $N = 20$, $p = 2$, $q = 5$, $d = 5$ - Basket Put option with different interest and borrowing rates
Non linear driver - Financial Benchmark

- We try this example in dimension $d = 10$
- The price and the $\delta$ are obtained in 5 minutes
Non Linear Driver & Path Dependent $\xi$

- The example is the following:
  - $d = 1$
  - $\xi = \sup_{0 \leq t \leq T} B_t$
  - $f(t, y, z) = \cos(y)$

- Comparison between $p = 2$ and $p = 3$ for $N = 20$ and $M = 10^5$
### Non Linear Driver & Path Dependent $\xi$

<table>
<thead>
<tr>
<th>iterations</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = 2$</td>
<td>1.237135</td>
<td>1.186691</td>
<td>1.195462</td>
<td>1.194256</td>
<td>14.06</td>
</tr>
<tr>
<td>$p = 3$</td>
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<td>1.183544</td>
<td>1.192367</td>
<td>1.191173</td>
<td>174.09</td>
</tr>
</tbody>
</table>

**Table**: Evolution of $Y_{0}^{q,p,N,M}$ w.r.t. Picard’s iterations, $M = 10^5$, $N = 20$ and CPU time

<table>
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<th>iterations</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$p = 2$</td>
<td>0.525273</td>
<td>0.459326</td>
<td>0.470069</td>
<td>0.469117</td>
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<tr>
<td>$p = 3$</td>
<td>0.523846</td>
<td>0.455827</td>
<td>0.466903</td>
<td>0.465939</td>
<td>174.09</td>
</tr>
</tbody>
</table>

**Table**: Evolution of $Z_{0}^{q,p,N,M}$ w.r.t. Picard’s iterations, $M = 10^5$, $N = 20$ and CPU time
Non Linear Driver & Path Dependent $\xi$

- Convergence in $M$

**Figure**: Evolution of $Y_{0}^{q,p,N,M}$ and $Z_{0}^{q,p,N,M}$ w.r.t. $q$ and $M$ when $N = 20$, $p = 2$, $\xi = \sup_{0 \leq t \leq T} B_{s}$, $f(t, y, z) = \cos(y)$.
Non Linear Driver & Path Dependent $\xi$

- Convergence in $N$

**Figure**: Evolution of $Y^{q,p,N,M}_0$ and $Z^{q,p,N,M}_0$ w.r.t. $N$ when $M = 10^5$, $p = 2$ - $\xi = \sup_{0 \leq t \leq T} B_s$, $f(t, y, z) = \cos(y)$
Conclusion

- The complexity of the algorithm is the following

\[ \mathcal{O}\left(K_{it} \times M \times p \times (N \times d)^{p+1}\right) \]

- This algorithm is
  - rather fast even when the dimension of the underlying BM is great
  - easy to implement

- We try it for quadratic BSDEs
  - It works only for \( \|\xi\|_\infty \) small enough
  - Not surprising w.r.t. R. Tevzadze’s result
Simulation of BSDEs and Wiener Chaos Expansions

Error Analysis
Splitting the error

- We want to control the error

\[ \mathcal{E} = \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t - Y_{t}^{q,p,N}|^2 + \int_{0}^{T} |Z_t - Z_{t}^{q,p,N}|^2 dt \right] \]

- Of course, we split the error into three parts

\[ \mathcal{E} = \mathcal{E}^q + \mathcal{E}^{q,p} + \mathcal{E}^{q,p,N} \]

\[ Y - Y_{q,p,N} = Y - Y^q + Y^q - Y_{q,p} + Y_{q,p} - Y_{q,p,N} \]

- A term is missing: \( Y_{q,p,N} - Y_{q,p,N,M} \) which comes from "Monte Carlo expectations"

- The first part is very easy:

\[ \mathcal{E}^q \leq C(T, L_f) 2^{-q}, \quad C(T, L_f) \text{ is explicit} \]
A lot of regularity

- The generator $f$ smooth: $f \in C_b^\infty$
- $\xi$ is smooth in the Malliavin sense: $\xi \in D^\infty$
  - Hölder continuity:
    $$|\mathbb{E}[D^{(r)}_{s_1,\ldots, s_r} \xi] - \mathbb{E}[D^{(r)}_{t_1,\ldots, t_r} \xi]| \leq K_\xi(|s_1 - t_1|^{\alpha \xi} + \cdots + |s_r - t_r|^{\alpha \xi})$$
  - The Malliavin derivatives belongs to $\bigcap L^p$
Example

- \( X \) solution to the SDE

\[
X_t = x + \int_0^t b(X_s) \, ds + \int_0^t \sigma(X_s) \, dB_s, \quad t \geq 0
\]

with \( b \) and \( \sigma \) \( C^\infty \) with bounded derivatives

- \( g : \mathbb{R}^n \rightarrow \mathbb{R} \, C_p^\infty \)

Regularity assumption true

\[
\xi = g(X_T) \quad \xi = g \left( \int_0^T X_s \, ds \right)
\]
Error Analysis Convergence Results

Error coming from a finite number of chaos

- We investigate now $\mathcal{E}^{q,p}$ related to $Y^q - Y^{q,p}$.

**Proposition**

*Under the regularity assumption*

$$\mathcal{E}^{q+1,p} \leq C_0 T (T + 1) L_f^2 \mathcal{E}^{q,p} + \frac{K_0}{p + 1}$$

where $K_0$ depends on the derivatives of $f$ and $\xi$ and $T$.

**Corollary**

We have

$$\mathcal{E}^{q,p} \leq \frac{A_0}{p + 1}, \quad A_0 = K_0 \frac{(C_0 T (T + 1) L_f^2)^q - 1}{C_0 T (T + 1) L_f^2 - 1}.$$
Error coming from a finite number of chaos

- We deduce from this result that

\[ \lim_{q \to +\infty} \lim_{p \to +\infty} Y^{q,p} = Y. \]

- When \( T \) is small, \( C_0 T (T + 1) L_f^2 < 1 \), \( A_0 \) does not depend on \( q \) and \( p \)

\[ \lim_{p \to +\infty} \lim_{q \to +\infty} Y^{q,p} = Y. \]

- The algorithm breaks the structure of BSDEs

- Let us write \( (Y^{q+1,p}, Z^{q+1,p}) \) in a backward way

\[
Y^{q+1,p}_t = C_p(\xi) + C_p \left( \int_0^T f(s, Y^{q,p}_s, Z^{q,p}_s) \, ds \right) - \int_0^T f(s, Y^{q,p}_s, Z^{q,p}_s) \, ds \\
+ \int_t^T f(s, Y^{q,p}_s, Z^{q,p}_s) \, ds - \int_t^T Z^{q+1,p}_s \, dB_s
\]
Error Analysis  
Convergence Results

Error coming from a finite number of chaos

- If \( \lim_{q \to \infty} Y_{q,p} \) exists, then one has

\[
Y_{\infty,p} = C_p(\xi) + C_p \left( \int_0^T f \left( s, Y_{\infty,p}, Z_{\infty,p} \right) ds \right) - \int_0^T f \left( s, Y_{\infty,p}, Z_{\infty,p} \right) ds \\
+ \int_t^T f(s, Y_{\infty,p}, Z_{\infty,p}) ds - \int_t^T Z_{\infty,p} dB_s
\]

- The terminal condition involves all the trajectory of the solution.
- We were able to solve this BSDE only when \( T \) is small
Error coming from the truncation of the basis

Proposition

Under the regularity assumption,

\[ \mathcal{E}^{q+1,p,N} \leq C_1 T(T + 1)L_f^2 \mathcal{E}^{q,p,N} + K_1 \left( \frac{T}{N} \right)^{2\alpha \xi^\wedge 1} \]

where \( C_1 \) is a scalar and \( K_1 \) depends on the derivatives of \( f \) and \( \xi \).

Moreover,

\[ \mathcal{E}^{q,p,N} \leq A_1 \left( \frac{T}{N} \right)^{2\alpha \xi^\wedge 1}, \quad A_1 = K_1 T(T + 1)e^T \frac{(C_1 T(T + 1)L_f^2)^q - 1}{C_1 T(T + 1)L_f^2 - 1} \]

- As before, the constant \( T(T + 1) \) appears
Convergence result

Proposition

Under the regularity assumption,

\[ \varepsilon \leq C 2^{-q} + \frac{A_0}{p + 1} + A_1 \left( \frac{T}{N} \right)^{2\alpha \xi} \wedge 1 \]

with

\[ A_0 = K_0 \frac{(C_0 T (T + 1)L_f^2)^q - 1}{C_0 T (T + 1)L_f^2 - 1} \]

\[ A_1 = K_1 T (T + 1) e^{T} \frac{(C_1 T (T + 1)L_f^2)^q - 1}{C_1 T (T + 1)L_f^2 - 1} \]

Corollary

In particular,

\[ \lim_{q \to \infty} \lim_{p \to \infty} \lim_{N \to \infty} Y_{q,p,N} = Y \]
**Perspectives**

- Try to weaken the regularity assumption
- Understand for large $p$ and $T$ the BSDE

\[
Y^\infty,p = C_p(\xi) + C_p \left( \int_0^T f(s, Y_s^{\infty,p}, Z_s^{\infty,p}) \, ds \right) - \int_0^T f(s, Y_s^{\infty,p}, Z_s^{\infty,p}) \, ds \\
+ \int_t^T f(s, Y_s^{\infty,p}, Z_s^{\infty,p}) \, ds - \int_t^T Z_s^{\infty,p} \, dB_s
\]

- This method works also for the dynamic programing equation

\[
Y_{t-h} = \mathbb{E}_{t-h}(Y_t) + hf(t-h, Y_{t-h}, Z_{t-h}), \quad Z_{t-h} = h^{-1} \mathbb{E}_{t-h}(Y_t(B_t - B_{t-h}))
\]

★ Numerical simulations are not so good
★ Error analysis
Thank you for your attention