Failure and Rescue in an Interbank Network

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Questions

- How can default and contagion be modelled in financial networks?
- Under which circumstances do banks have incentives to bail-out defaulting banks?
- What are resolution mechanisms of financial distress?
An Interbank Network

![Interbank Network Diagram]

1. Bank 1 to Bank 2: 4.94
2. Bank 1 to Bank 3: 2.47
3. Bank 1 to Bank 6: 2.79
4. Bank 2 to Bank 3: 6.21
5. Bank 2 to Bank 4: 2.47
6. Bank 3 to Bank 4: 5.59
7. Bank 4 to Bank 5: 8
8. Bank 4 to Bank 6: 12
9. Bank 5 to Bank 6: 12.5
10. Bank 6 to Bank 1: 1.4
11. Bank 6 to Bank 3: 11.51

Luitgard A. M. Veraart (LSE)  Failure and Rescue in an Interbank Network  October 2012 3 / 27
The Setting

- Market with \( n \) banks with indices in \( \mathcal{N} := \{1, \ldots, n\} \); nodes in a network.

- **The liabilities matrix** is \( L \in \mathbb{R}^{n \times n} \), where the \( ij^{th} \) entry \( L_{ij} \) represents the nominal liability of bank \( i \) to bank \( j \). Assumption: \( L_{ij} \geq 0 \ \forall \ i, j \) and \( L_{ii} = 0 \ \forall \ i \).

- The **total nominal obligations** of bank \( i \) to all other banks in the system are given by \( \bar{L}_i = \sum_{j=1}^{n} L_{ij} \).

- The **relative liabilities matrix** \( \Pi \in \mathbb{R}^{n \times n} \) is defined by
  \[
  \pi_{ij} := \begin{cases} 
  L_{ij}/\bar{L}_i & \text{if } \bar{L}_i > 0, \\
  0 & \text{otherwise.}
  \end{cases}
  \]

- We denote by \( e_i \geq 0 \) the **net assets** of bank \( i \) from sources outside the banking system.
Consider network with liabilities matrix $L$ and net external assets $e$.

Check for each bank $i \in \mathcal{N}$ whether
\[
\sum_{j=1}^{n} \pi_{ji} \bar{L}_j + e_i - \bar{L}_i \geq 0.
\]

If this is $< 0$ for some banks, some banks are bankrupt.

Assumptions in case of default:

- **Default costs:**
  Fraction of net external assets realized on liquidation: $\alpha \in (0,1]$, 
  Fraction of interbank assets realized on liquidation: $\beta \in (0,1]$.

- **Clearing:** Clearing mechanism determines the payments between banks.
The Clearing Mechanism

- **Limited liabilities**: Nodes never pay more than available cash flow.
- **Priority of debt claims over equity**: Paying off the liabilities $L_{ij}$ has priority, even if net assets $e_i$ have to be used.
- **Proportionality**: If default occurs the defaulting bank pays all claimant banks in proportion to the size of their nominal claims on the assets of the defaulting bank.

A **clearing vector** for the financial system $(L, e, \alpha, \beta)$ is a vector $L^* \in [0, \bar{L}]$ such that

$$L^* = \Phi(L^*),$$

$$\Phi(L)_i := \begin{cases} \bar{L}_i, & \text{if } \bar{L}_i \leq e_i + \sum_{j=1}^{n} L_j \pi_{ji}, \\ \alpha e_i + \beta \sum_{j=1}^{n} L_j \pi_{ji}, & \text{else}. \end{cases}$$
Theorem (Existence of Clearing Vectors)

Let $\alpha, \beta \in (0, 1]$. For every financial system $(L, e, \alpha, \beta)$ there exist clearing vectors $L^*$ and $L_*$ such that for any clearing vector $L$, we have 

$$L_* \leq L \leq L^*.$$
Greatest Clearing Vector Algorithm (GA)

constructs sequence \((\Lambda^{(\nu)})\):

1. Set \(\nu = 0\), \(\Lambda^{(0)} := \bar{L}\) and \(\mathcal{I}_{-1} := \emptyset\).

2. For all nodes \(i\) compute \(v_i^{(\nu)} := \sum_{j=1}^{n} \Lambda_j^{(\nu)} \pi_{ji} + e_i - \bar{L}_i\).

3. Insolvent banks: \(\mathcal{I}_\nu = \{i : v_i^{(\nu)} < 0\}\), solvent banks: \(S_\nu = \{i : v_i^{(\nu)} \geq 0\}\).

4. If \(\mathcal{I}_\nu \equiv \mathcal{I}_{\nu-1}\) terminate the algorithm.
   Otherwise set \(\Lambda_j^{(\nu+1)} := \bar{L}_j\) \(\forall j \in S_\nu\), solve
   \[
x_i = \alpha e_i + \beta \left\{ \sum_{j \in S_\nu} \bar{L}_j \pi_{ji} + \sum_{j \in \mathcal{I}_\nu} x_j \pi_{ji} \right\} \quad \forall i \in \mathcal{I}_\nu
   \]
   and set \(\Lambda_i^{(\nu+1)} := x_i\) for \(i \in \mathcal{I}_\nu\).

5. Set \(\nu \to \nu + 1\) and go back to 2.

When the algorithm has terminated, the vector \(\Lambda^{(\nu)}\) is a clearing vector.
Example I

$$e = \begin{pmatrix} 1 \\ 1/2 \\ 3/2 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}; \quad \bar{L} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. $$
Net assets are reduced to $\gamma e$, $\gamma \in [0, 1]$. 
Example: $\gamma = \frac{1}{6}$. Define new net assets $e := \left( \frac{1}{6}, \frac{1}{12}, \frac{1}{4} \right)^\top$. $\nu = 0$, $\Lambda^{(0)} = \overline{L}$, $\nu^{(0)} = \left( \frac{1}{6} + \frac{1}{2} - \frac{1}{2}, \frac{1}{12} + \frac{1}{2}, \frac{1}{4} - \frac{1}{2} \right)^\top$. $I_0 = \{3\}$, $S_0 = \{1, 2\}$. 
Example IV

\[ I_0 = \{3\}, \ S_0 = \{1, 2\}. \]

\[ \Lambda_1^{(1)} = \bar{L}_1 = \frac{1}{2}, \ \Lambda_2^{(1)} = \bar{L}_2 = 0, \ \Lambda_3^{(1)} = x_3 \] where

\[ x_3 = \alpha e_3 + \beta \left\{ \sum_{j \in S_0} \bar{L}_j \pi_3 + \sum_{j \in I_0} x_j \pi_3 \right\} = \alpha \frac{1}{4} + \beta \left\{ \frac{1}{2} \cdot 0 + 0 + 0 \right\} = \frac{\alpha}{4}. \]
Example V

\[ \nu := 1. \quad \alpha = \frac{1}{2}. \quad \nu^{(1)} = \left( \frac{1}{6} + \frac{1}{8} - \frac{1}{2}, \frac{1}{12} + \frac{1}{2}, \frac{1}{4} - \frac{1}{2} \right)^\top, \]

\[ \mathcal{I}_1 = \{1, 3\}, \quad S_1 = \{2\}. \]

\[ \Lambda^{(2)}_1 = x_1, \quad \Lambda^{(2)}_2 = \bar{L}_2 = 0, \quad \Lambda^{(1)}_3 = x_3, \] where

\[ x_1 = \alpha e_1 + \beta \left\{ \sum_{j \in S_0} \bar{L}_j \pi_j 1 + \sum_{j \in \mathcal{I}_0} x_j \pi_j 1 \right\} = \frac{1}{12} + \beta \{0 + x_3 + x_1 \cdot 0\} = \frac{1}{12} + \beta x_3. \]

\[ x_3 = \alpha e_3 + \beta \left\{ \sum_{j \in S_0} \bar{L}_j \pi_j 3 + \sum_{j \in \mathcal{I}_0} x_j \pi_j 3 \right\} = \frac{1}{8} + \beta \{\frac{1}{2} \cdot 0 + x_1 \cdot 0 + x_3 \cdot 0\} = \frac{1}{8}. \]
Example VI

\[ \nu := 2. \quad \nu^{(2)} = \left( \frac{1}{6} + \frac{1}{8} - \frac{1}{2}, \frac{1}{12} + \frac{1}{12} + \frac{\beta}{8}, \frac{1}{4} - \frac{1}{2} \right)^\top, \]
\[ \mathcal{I}_2 = \{1, 3\} = \mathcal{I}_1. \]
STOP!

\[ L^* = \Lambda^{(2)} = \left( \frac{1}{12} + \frac{\beta}{8}, 0, \frac{1}{8} \right)^\top. \]
Consequences

- The greatest clearing vector algorithm (GA) produces a sequence of vectors $\Lambda^{(\nu)}$ decreasing in at most $n$ iterations to the greatest clearing payment vector.

- We call the set $I_\nu$ the level-$\nu$ insolvency set.

- The level-0 insolvency set is the set of those banks which would default even if all other banks paid their obligations in full.

- The level-$\nu$ insolvency set is the set of all those banks which would not be able to meet their obligations if all the level-$(\nu - 1)$ insolvent banks were to default.

- The insolvency sets $I_\nu$ trace the spread of default through the financial system.
Bank Merger

- Given a financial system \((L, e, \alpha, \beta)\) with banks of indices \(\mathcal{N}\).

- After a merger of all banks in \(B \subset \mathcal{N}\) one obtains a new financial system \((\tilde{L}, \tilde{e}, \alpha, \beta)\) indexed by \(\tilde{\mathcal{N}} := \{0\} \cup (\mathcal{N} \setminus B)\).

- We assume that a merger is associated with costs. We model costs from merger in terms of a vector \(\kappa \in \mathbb{R}^n_+\). If bank \(i\) is involved in a merger, costs of size \(\kappa_i\) occur.

- **New liabilities matrix** \(\tilde{L}\): The liabilities of the merging banks to the other (non merging) banks in the network are added up. The liabilities to those banks which merge are cancelled.

- **New net assets** \(\tilde{e}\): The net assets of the merged bank are the sum of the net assets of the banks that merged minus the costs for merger, \(\sum_{i \in B} \kappa_i\).
Example of Merger - Initial Network

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Failure and Rescue in an Interbank Network
October 2012 17 / 27
Example of Merger - Change Liabilities and Net Assets
Example of Merger - New Network

\[ \tilde{e}_0 = e_1 + e_4 \]

\[ \tilde{l}_{01} = l_{12} \]
\[ \tilde{l}_{20} = l_{21} + l_{24} \]
\[ \tilde{l}_{03} = l_{13} + l_{43} \]
\[ \tilde{e}_2 = e_2 \]
\[ \tilde{l}_{32} = l_{32} \]
\[ \tilde{e}_3 = e_3 \]
Rescue Consortium

Let \((\mathbf{L}, \tilde{\mathbf{e}}, \alpha, \beta)\) be a financial system where the level-0 insolvency set \(\mathcal{I}_0\) is non-empty and let \(L^*\) be the greatest clearing vector.

- The value \(\mathcal{V}\) of the banks is defined as

\[
\mathcal{V}(L^*, \tilde{e})_i := (\Pi^\top L^* + \tilde{e} - L^*)_i I\{L^*_i \geq \bar{L}_i\}.
\]

- The bailout costs are \(\sum_{j \in \mathcal{I}_0} \delta_j\), with \(\delta := \max\{0, - (\Pi^\top \bar{L} + \tilde{e} - \bar{L})\}\).

- Let

\[
\tilde{\mathcal{V}} := \max\{0, \Pi^\top \bar{L} + \tilde{e} - \bar{L}\},
\]

\[
\Delta \mathcal{V} := \tilde{\mathcal{V}} - \mathcal{V}(L^*, \tilde{e}).
\]

A rescue consortium is a set \(A \subseteq \mathcal{N} \setminus \mathcal{I}_0\) such that:

1. \(\sum_{i \in A} \Delta \mathcal{V}_i > \sum_{j \in \mathcal{I}_0} \delta_j + \sum_{k \in A \cup \mathcal{I}_0} \kappa_k\) (rescue incentive),

2. \(\sum_{i \in A} \tilde{\mathcal{V}}_i > \sum_{j \in \mathcal{I}_0} \delta_j + \sum_{k \in A \cup \mathcal{I}_0} \kappa_k\) (rescue ability).
Rescue Incentive and Rescue Ability

Theorem

1. Every rescue consortium which has an incentive to rescue the failing banks also has the ability to rescue the failing banks.

2. Suppose that the set of banks at risk of contagious default \( \mathcal{R} := \bigcup_{\nu} \mathcal{I}_\nu \backslash \mathcal{I}_0 \) is non-empty, and suppose, further that some subset \( A \subseteq \mathcal{R} \) is able to rescue the failing banks. Then \( A \) also has an incentive to rescue the failing banks.

Definition (Rescued Financial System)

Let \((L, \tilde{e}, \alpha, \beta)\) be a financial system in which the level-0 insolvency set \( \mathcal{I}_0 \) is non-empty, and suppose that a rescue consortium defined by a set of indices \( A \) exists. Then the rescued financial system is the financial system obtained by a merger of all banks in \( \mathcal{I}_0 \cup A \).
Absence of Rescue Consortium

Theorem

Consider a financial system \((L, e, \alpha, \beta)\) in which all banks are initially solvent.

Suppose the assets \(e\) are reduced to \(\tilde{e}\), with \(\tilde{e}_i \leq e_i \forall i\) with the result that at least one bank becomes level-0 insolvent.

Suppose that \(\alpha = \beta = 1\).

Then no group of banks in the network has an incentive to rescue the insolvent bank(s).
Presence of Rescue Consortium

**Theorem**

Let \((L, \tilde{e}, \alpha, \beta)\) be a financial system with \(\alpha, \beta \in [0, 1)\). Suppose that \(\mathcal{I}_0\) is a proper subset of \(\mathcal{N}: \emptyset \subsetneq \mathcal{I}_0 \subsetneq \mathcal{N}\). Let \(L^*\) be the corresponding greatest clearing vector and let \(\kappa\) be the vector describing the costs for merger. Suppose

\[
\sum_{i=1}^{n} \left( (1 - \alpha)\tilde{e}_i + (1 - \beta) \sum_{j=1}^{n} L^*_j \pi_{ji} \right) \mathbb{I}_{\{L^*_i < \bar{L}_i\}} > \sum_{k=1}^{n} \kappa_k.
\]

Then there exists a rescue consortium.
Possible Resolution Mechanisms for Bank Rescues I

Suppose that the banks in $\mathcal{I}_1$ are capable of mounting a rescue. The regulator should be empowered to compel that group of banks to rescue the banks in $\mathcal{I}_0$. Main ideas for different mechanisms:

1. Each bank in $\mathcal{I}_1$
   - contributes to bailout costs in proportion to losses $\Delta V$ that they would experience if default were to occur,
   - receives shares in rescued banks in proportion to their contribution.

2. Each bank $i$ in $\mathcal{I}_1$ gives a sealed bid to the regulator containing the fraction $\alpha_i$ of the bailout costs which it was willing to take:
   - If $\sum_{i \in \mathcal{I}_1} \alpha_i \geq 1$, banks in $\mathcal{I}_1$ are allocated fractions of the defaulting banks assets and liabilities proportional to their bids $\alpha_i$. 
• \( \sum_{i \in \mathcal{I}_1} \alpha_i < 1 \), then each bank in \( \mathcal{I}_1 \) contributes to the bailout proportionally to its potential losses \( \Delta V \) as in mechanism (1) above, but receives a fraction of the defaulting banks proportional to its bid.

3. Allow the regulator to seize the assets of any failing bank, which would then pay out nothing to any bank to which it owed money.

• Assets could be used to compensate depositors, with any not used in this way being held by the government.

• Gives other banks a very strong incentive to mount a rescue.
Conclusion

- Modelled contagion in financial networks.
- Derived conditions for existence of a rescue consortium.
- Default costs are necessary for the existence of a rescue consortium.
- Role of regulator and bail-out decisions.
- Model can be used for stress testing a given network.
- Could include random shocks etc.

Lemma

Consider a financial system \((\mathbf{L}, e, \alpha, \beta)\) in which all banks are initially solvent. Suppose that the assets \(e\) are reduced to \(\tilde{e}\), with \(\tilde{e}_i \leq e_i \forall i\) such that some banks have become insolvent: \(\mathcal{I}_0 \neq \emptyset\). Let \(L^*\) be the greatest clearing vector in \((\mathbf{L}, \tilde{e}, \alpha, \beta)\). Then

\[
0 \leq \sum_{i=1}^{n} \left( \mathcal{V} \left( \mathbf{L}, e \right)_i - \mathcal{V} \left( L^*, \tilde{e} \right)_i \right) \\
= \sum_{i=1}^{n} (e_i - \tilde{e}_i) + \sum_{i=1}^{n} \left( (1 - \alpha)\tilde{e}_i + (1 - \beta) \sum_{j=1}^{n} L^*_j \pi_{ji} \right) \mathbb{I}_{\{L^*_i < \bar{L}_i\}}.
\]
Theorem

Let \((L, \tilde{e}, \alpha, \beta)\) be a financial system with \(\alpha, \beta \in [0, 1)\). Suppose that \(I_0\) is a proper subset of \(\mathcal{N}: \emptyset \subsetneq I_0 \subsetneq \mathcal{N}\). Suppose that the costs for merger are \(\kappa \equiv 0\).

Let \(L^*\) be the corresponding greatest clearing vector and suppose that there exists a bank \(k\) such that \(L^*_k < \bar{L}_k\) which satisfies at least one of the following two conditions:

1. \(\tilde{e}_k > 0\),

2. there exists \(j \neq k\) such that \(L^*_j \pi_{jk} > 0\).

Then there exists a rescue consortium.
Consider financial system \((\mathbf{L}, e, \alpha, \beta)\) with \(\mathcal{I}_0 = \emptyset\). Reduce net assets to \(\gamma e, \gamma \in [0,1]\) and consider

- the relative losses due to default:

\[
\gamma \mapsto \lambda(\gamma) := \frac{\sum_{i=1}^{n} e - v(\gamma)}{\sum_{i=1}^{n} e}
\]

The function \(\lambda\) measures the difference between the initial value of the system and the value of the stressed system divided by the initial value of the system.

- the proportion of defaulting banks:

\[
\gamma \mapsto \eta(\gamma) := \frac{|\{i \in \mathcal{N} \mid L_i^*(\gamma) < \bar{L}_i\}|}{|\mathcal{N}|}
\]

where \(L^*(\gamma)\) is the greatest clearing vector if the external assets are given by \(\tilde{e} = \gamma e\).
Assessing and Controlling Contagion Risk

Plot of $\lambda$ and $\eta$ for the asymmetric network with 6 banks, $\beta = 0.9$. 

![Plot of $\lambda(\gamma, \alpha)$ and $\eta(\gamma, \alpha)$ for an asymmetric network with 6 banks, $\beta = 0.9$.]