BSDEs with terminal conditions that have bounded Malliavin derivative
...and driver that have arbitrary growth

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BSDEs, Numerics, and Finance
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Motivation

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\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s \]

Let \( W \) be \( n \) dimensional Brownian motion on \( (\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}) \) where \( (\mathcal{F}_t) \) be the filtration generated by \( W \). \( \xi \) is \( \mathcal{F}_T \) measurable \( \mathbb{L}^2(d\mathbb{P}) \) random variable. \( f \) is progressively measurable and \( f(\cdot, 0, 0) \in \mathbb{H}^2 \).

- (Pardouix and Peng) \( f \) is globally Lipschitz in \((y, z) \Rightarrow \exists!(Y, Z) \in \mathbb{S}^2 \times \mathbb{H}^2(\mathbb{R}^n)\).

- (Kobylanski) \( \xi \) is bounded and \( f(t, y, z) = a_0(t, y, z) y + f_0(t, y, z) \) where \( a_0 \) is bounded and \( f_0 \) has growth like \( b + \rho(|y|)|z|^2 \) for an increasing function \( \rho \Rightarrow \exists(Y, Z) \in \mathbb{H}^\infty \times \mathbb{H}^2 \).

Can we assume superquadratic growth in \( z \) with/without superlinear growth in \( y \)?

\( \Rightarrow \) Under “regularity” condition on \( \xi \), yes.
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Previous Works

- Quadratic growth in $z$
  - Kobylanski (2000): bounded $\xi$
  - Briand & Hu (2006, 2008), Delbaen et al. (2011): unbounded $\xi$
- Superquadratic growth in $z$
  - Delbaen et al. (2010): convex driver under Markovian framework and bdd $\xi$.
  - Cheridito & Stadje (2011): convex driver and Lipschitz $\xi$.
- Superlinear growth in $y$
  - Lepeltier & San Martin (1997): $|y|\sqrt{\log|y|}$ like growth.
  - Briand & Carmona (2000): monotonicity condition
- Quadratic in $z$ and Polynomial in $y$
  - Kobylanski (2000): bounded $\xi$
  - Frei & Dos Reis (2012): bounded $\xi$ and
    $|f(s, y, z)| \leq C(1 + |y| + (1 + |y|^k)|z|^2)$
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Main Theorem

We assume the following conditions.

(A1) $\xi \in \mathbb{D}^{1,2}$ with $|D_i^r \xi| \leq C_i \, dr \otimes d\mathbb{P}$ a.e.

(A2) $f(\cdot, 0, 0) \in \mathbb{H}^4$

(A3) For a nondecreasing function $\rho$,

\[
|f(s, y, z) - f(s, y', z)| \leq B|y - y'|
\]
\[
|f(s, y, z) - f(s, y, z')| \leq \rho(|z| \lor |z'|)|z - z'|
\]

(A4) $f(\cdot, y, z) \in \mathbb{L}^{1,2}_{a}$ and $|D_r f(s, y, z)| \leq \phi_i(s)$, $dr \otimes d\mathbb{P}$-a.e. where $\phi_i$ are Borel measurable with $\int_0^T \phi_i^2(t) dt < \infty$

(A5) $|D_r f(s, y, z) - D_r f(s, y', z')| \leq K_{r,s}^R (|y - y'| + |z - z'|)$ for all $|z|, |z'| \leq R$ where for a.a $r$, $K_{r,s}^R \in \mathbb{H}^4$ such that

$\int_0^T \|K_{r,s}^R\|_{\mathbb{H}^4}^4 \, dr < \infty$

Then, our BSDE has a unique solution $(Y, Z) \in S^4 \times \mathbb{H}^\infty(\mathbb{R}^n)$. Moreover, the bound of $Z_t^i$ is \( C_i + \int_t^T \phi_i(s) ds \) $e^{B(T-t)}$.

(cf) If $K_{r,s}^R$ is bounded, we can replace (A2) to $f(\cdot, 0, 0) \in \mathbb{H}^2$ and $Y \in S^2$ instead of $S^4$. 
Main Theorem

We assume the following conditions.

(A1) $\xi \in \mathbb{D}_{1,2}$ with $|D^i_r\xi| \leq C_i \, dr \otimes d\mathbb{P}$ a.e.

(A2) $f(\cdot, 0, 0) \in \mathbb{H}^4$

(A3) For a nondecreasing function $\rho$,
$$
|f(s, y, z) - f(s, y', z)| \leq B|y - y'| \\
|f(s, y, z) - f(s, y, z')| \leq \rho(|z| \lor |z'|)|z - z'|
$$

(A4) $f(\cdot, y, z) \in \mathbb{L}_{a,2}^1$ and $|D_r f(s, y, z)| \leq \phi_i(s)$, $dr \otimes d\mathbb{P}$-a.e. where $\phi_i$ are Borel measurable with $\int_0^T \phi_i^2(t)dt < \infty$

(A5) $|D_r f(s, y, z) - D_r f(s, y', z')| \leq K^R_{r,s} (|y - y'| + |z - z'|)$ for all $|z|, |z'| \leq R$ where for a.a $r$, $K^R_{r,s} \in \mathbb{H}^4$ such that
$$
\int_0^T \|K^R_{r,s}\|_{\mathbb{H}^4}^4 \, dr < \infty
$$

Then, our BSDE has a unique solution $(Y, Z) \in \mathbb{S}^4 \times \mathbb{H}^\infty(\mathbb{R}^n)$.

Moreover, the bound of $Z^i_t$ is
$$
(C_i + \int_t^T \phi_i(s)ds) \, e^{B(T-t)}
$$

(cf) If $K^R_{r,s}$ is bounded, we can replace (A2) to $f(\cdot, 0, 0) \in \mathbb{H}^2$ and
$Y \in \mathbb{S}^2$ instead of $\mathbb{S}^4$. 
Main Corollary

If we assume stronger solution than (A1) and (A2), we can generalize (A3).

(A1') (A1) and $|\xi| \leq E$

(A2') $|f(s, y, z)| \leq G(1 + |y|) + \psi(|y|, |z|)|z|$ for nondecreasing function $\psi$

(A3') For nondecreasing functions $\rho_1$ and $\rho_2$, 
- $|f(s, y, z) - f(s, y', z)| \leq \rho_1(|y| \vee |y'|)|y - y'|$
- $|f(s, y, z) - f(s, y, z')| \leq \rho_2(|y|, |z| \vee |z'|)|z - z'|$

(A5') $|D_r f(s, y, z) - D_r f(s, y', z')| \leq K^R_{r,s}(|y - y'| + |z - z'|)$ for all $|y|, |y'|, |z|, |z'| \leq R$ where for a.a $r$, $K^R_{r,\cdot} \in \mathbb{H}^4$

In addition, we assume (A4). Then, our BSDE has a unique solution $(Y, Z) \in \mathbb{S}^\infty \times \mathbb{H}^\infty(\mathbb{R}^n)$. Moreover,
- $|Y_t| \leq (E + 1)e^{G(T-t)} - 1 := d(t)$ for all $t$ a.s.
- $|Z_t^i| \leq \left(C_i + \int_t^T \phi_i(s)ds\right)e^{\rho_1(d(t))(T-t)} =: e_i(t)$ for almost all $t$ a.s.

$$\text{ex} f(y, z) = |z|^3 + \frac{|z|}{1 + |z|}|y|^2$$
Main Corollary

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(A1') (A1) and $|\xi| \leq E$

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\[
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\]
\[
|f(s, y, z) - f(s, y, z')| \leq \rho_2(|y|, |z| \lor |z'|)|z - z'|
\]

(A5') $|D_r f(s, y, z) - D_r f(s, y', z')| \leq K_{r,s}^R (|y - y'| + |z - z'|)$ for all $|y|, |y'|, |z|, |z'| \leq R$ where for a.a $r$, $K_{r,s}^R \in H^4$

In addition, we assume (A4). Then, our BSDE has a unique solution $(Y, Z) \in S^\infty \times H^\infty(\mathbb{R}^n)$. Moreover,

$|Y_t| \leq (E + 1)e^{G(T-t)} - 1 =: d(t)$ for all $t$ a.s.

$|Z_t^i| \leq \left(C_i + \int_t^T \phi_i(s)ds\right)e^{\rho_1(d(t))(T-t)} =: e_i(t)$ for almost all $t$ a.s.

ex) $f(y, z) = |z|^3 + \frac{|z|}{1+|z|}|y|^2$
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1 Assume that (A1-(A4) hold and (A5) holds with \( K \) independent of \( R, f \) is differentiable in \( y \) and \( z \), and \( \rho \) being constant. Then,

**Theorem (El Karoui et al(1997))**

We have a unique solution \((Y, Z)\) in \( S^4(\mathbb{R}) \times H^4(\mathbb{R}^n)\). Moreover, \((Y, Z) \in L_{a}^{1,2}(\mathbb{R}^{n+1})\) and for all \( i = 1, \ldots, n \),

\[
(D^i_r Y_t, D^i_r Z_t) = (U^r_t, V^r_t) \, dr \otimes dt \otimes d\mathbb{P}\text{-a.e. and } Z^i_t = U^t_t \, dt \otimes d\mathbb{P}\text{-a.e.}
\]

where \( U^r_t = 0 \) and \( V^r_t = 0 \) for \( 0 \leq t < r \leq T \), and, for each fixed \( r \),

\((U^r_t, V^r_t)_{r \leq t \leq T}\) is the unique pair in \( S^2(\mathbb{R}) \times H^2(\mathbb{R}^n)\) solving the BSDE

\[
U^r_t = D^i_r \xi + \int_t^T [\partial_y f(s, Y_s, Z_s) U^r_s + \partial_z f(s, Y_s, Z_s) V^r_s + D^i_r f(s, Y_s, Z_s)] ds - \int_t^T V^r_s dW_s.
\]
2 Using Ito formula on $e^{Bt} |U_t|^2$, we can bound $D_r Y_t$ and $Z_t$.
3 Show the solution exists uniquely for small time with $Z$ bounded by big enough $S. \leq$ cutoff argument.
4 By comparison theorem, we get better bound for $Z$ which is not blowing up.
5 Iterate the procedure to time 0.
6 Using mollification, we can show the claim for Lipschitz driver.
7 Generalize to $K^R$ using the explicit bound of $Z$.
8 Under (A1')-(A3'), (A4) and (A5'), by comparison theorem, we can bound $Y$ and prove the corollary.
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Markovian BSDEs and semilinear parabolic PDEs

For some $B > 0$, consider

$$Y_{s}^{t,x} = h(X_{T}^{t,x}) + \int_{s}^{T} g(r, X_{r}^{t,x}, Y_{r}^{t,x}, Z_{r}^{t,x})dr - \int_{s}^{T} Z_{r}^{t,x}dW_{r} \quad (1)$$

where

$$X_{s}^{t,x} = x + \int_{t}^{s} b(r, X_{r}^{t,x})dr + \int_{t}^{s} \sigma(r)dW_{r}$$

where $|b(s, x) - b(s, y)| \leq B|x - y|$, $|b(s, x)| \leq B(1 + |x|)$, and $|\sigma| \leq B$.

(B1) $h : \mathbb{R}^m \rightarrow \mathbb{R}$ is Lipschitz with coefficient $B$.

(B2) $\int_{0}^{T} |g(s, 0, 0, 0)|^2 ds < \infty$.

(B3) For a nondecreasing function $\rho$,

$$|g(s, x, y, z) - g(s, x, y', z)| \leq B|y - y'|$$

$$|g(s, x, y, z) - g(s, x, y, z')| \leq \rho (|z| \vee |z'|) |z - z'|$$

(B4) $|g(s, x, y, z) - g(s, x', y, z)| \leq B|x - x'|$

(B5) $|g(s, x, y, z) - g(s, x', y, z) - g(s, x, y', z') + g(s, x', y', z')| \leq B^{R}|x - x'| (|y - y'| + |z - z'|)$ for all $|z|, |z'| \leq R$.
(B1') (B1) with $|h| \leq B$

(B2') $|g(s, x, y, z)| \leq B(1 + |y|) + \psi(|y|, |z|)|z|$

(B3') For nondecreasing functions $\rho_1$ and $\rho_2$,

$$|g(s, x, y, z) - g(s, x, y', z)| \leq \rho_1(|y| \vee |y'|)|y - y'|$$

$$|g(s, x, y, z) - g(s, x, y, z')| \leq \rho_2(|y|, |z| \vee |z'|)|z - z'|$$

(B5') $|g(s, x, y, z) - g(s, x', y, z) - g(s, x, y', z') + g(s, x', y', z')| \leq B^R|x - x'|(|y - y'| + |z - z'|)$ for all $|y|, |y'|, |z|, |z'| \leq R$

**Theorem**

If (B1)-(B5) hold, then BSDE (1) has a unique solution such that $X^{t,x}$ is a square integrable strong solution and $(Y^{t,x}, Z^{t,x}) \in S^4 \times H^\infty(\mathbb{R}^n)$.

**Theorem**

If (B1')-(B3'),(B4), and (B5') hold, then BSDE (1) has a unique solution such that $X^{t,x}$ is a square integrable strong solution, $(Y^{t,x}, Z^{t,x}) \in S^\infty \times H^\infty(\mathbb{R}^n)$. 
Sketch of Proof

It is easy to check conditions of main theorem and corollary using the following two lemmas.

**Lemma**

\[ X^{t,x} \in D^{1,2} \text{ with } |D_r X^{t,x}| \leq B e^{BT}. \]

**Proof.**

Differentiability is proved in the reference and we can use Gronwall’s inequality to Malliavin derivative of above SDE of \( X^{t,x} \).

**Lemma (Proposition 1.1.4 of Nualart(2006))**

If \( \phi \) is \( L \)-Lipschitz and \( X \in D^{1,2} \), then \( \phi(X) \in D^{1,2} \) and there exists random vector \( R \) bounded by \( L \) such that \( D\phi(X) = R(DX) \).
Viscosity Solution

Theorem

Assume either (B1)-(B5) or (B1’)-(B3’), (B4), (B5’). Then, $u(t,x) := Y^{t,x}_t$ is a unique viscosity solution of

$$u_t(t,x) + L_t u(t,x) + g(t,x,u(t,x), (\nabla u\sigma)(t,x)) = 0 \quad (2)$$

$$u(T,x) = g(x)$$

where

$$L_t = \frac{1}{2} \sum_{i,j} (\sigma\sigma^T)_{i,j}(t) \partial x_i \partial x_j + \sum_i b_i(t,x) \partial x_i.$$ 

Proof.

By standard result, it is known $u$ is a viscosity solution. By Ishii and Lions(1990), under our conditions, viscosity solution is unique.
Assume either (i) or (ii) holds.

(i) Assume (B1)–(B5). Also, suppose that $b(s, x), h(x)$, and $g(s, x, y, z)$ are $C^3$ in $(x, y, z)$ and they have bounded first order, second order, and third order derivatives in

$$\{(s, x, y, z) \in [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n : |z| \leq M_z\}$$

where $M_z$ is the bound of $Z^{t,x}$ in our previous theorem.

(ii) Assume (B1’)–(B3’), (B4), (B5’). Also, suppose that $b(s, x), h(x)$, and $g(s, x, y, z)$ are $C^3$ in $(x, y, z)$ and they have bounded first order, second order, and third order derivatives in

$$\{(s, x, y, z) \in [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n : |y| \leq M'_y, |z| \leq M'_z\}$$

where $M'_y$ and $M'_z$ are the bounds of $Y^{t,x}$ and $Z^{t,x}$ in our previous theorem, respectively.
Theorem

Then, $u(t, x)$ in previous theorem is actually of class $C^{1,2}([0, T] \times \mathbb{R}^m; \mathbb{R})$ which solves (2). Moreover, if (i) holds, $\nabla u(t, x)\sigma(t)$ is bounded for almost every $t$. If (ii) holds, then $u(t, x)$ is bounded for all $t$ and $\nabla u(t, x)\sigma(t)$ is bounded for almost every $t$.

Proof.

Apply the standard result in El Karoui et al.(1997) with appropriate bound.

cf) When $g(z) = a|z|^d$ with $d > 1$, these results closely related with Amour and Ben-Artzi(1998), Gilding et al.(2003) which deals with classical solutions. For classical solution, Amour and Ben-Artzi assumed bounded terminal condition with $C^2_b$. Gilding et al. assumed the boundedness of terminal condition and fixed $a = 1$. 
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We provide a sufficient condition for (A1).

**Definition**

We call a random variable $\xi$ Lipschitz continuous in the Brownian motion $W$ with constants $C_1, \ldots, C_n \in \mathbb{R}^+$ if $\xi = \varphi(W)$ for a function $\varphi : C([0, T]; \mathbb{R}^n) \to \mathbb{R}$ satisfying

$$|\varphi(v) - \varphi(w)| \leq \sup_{0 \leq t \leq T} \sum_{i=1}^{n} C_i |v^i(t) - w^i(t)|.$$ 

**Theorem**

Let $\xi$ be Lipschitz continuous in $W$ with constants $C_1, \ldots, C_n \in \mathbb{R}^+$. Then $\xi \in D^{1,2}$ and $|D^i_\xi| \leq C_i \, dt \otimes d\mathbb{P}$-a.e. for all $i = 1, \ldots, n$. Converse is not TRUE.
Example

Assume $T = n = 1$. Define

$$g(t) := \sum_{k=1}^{\infty} (-1)^{k-1} 2^k 1_{\{1-2^{-1-k} < t \leq 1-2^{-k}\}}, \quad h(t) := \int_0^t g(s)ds,$$

and set

$$\xi := \int_0^1 h(t)dW_t.$$

Then $D\xi = h$ is bounded by 1.

On the other hand, it follows from integration by parts that

$$\xi = \lim_{k \to \infty} \int_0^{1-2^{-2k}} g(t) W_t dt.$$
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Let $\mathcal{O}$ be a bounded open subset in $\mathbb{R}^m$ with smooth boundary. Let the forward process in FBSDE (1) is changed to

$$X_{s}^{t,x} = x + \int_{t}^{s} b(X_{r}^{t,x}) \, dr + \int_{t}^{s} \sigma(X_{r}^{t,x}) \, dW_r + \int_{t}^{s} n(X_{r}^{t,x}) \, d\kappa_{r}^{t,x}$$

$$\kappa_{s}^{t,x} = \int_{t}^{s} 1_{\{X_{r}^{t,x} \in \partial \mathcal{O}\}} \, d\kappa_{r}^{t,x}, \quad \kappa_{.}^{t,x} \text{ is increasing}$$

**Example**

Assume $\mathcal{O}$ is an open bounded box, $b$ is Lipschitz, $\sigma$ is a constant. Then, $X^{t,x}$ is Lipschitz in underlying Brownian motion. Therefore, $X^{t,x} \in D^{1,2}$ and $DX^{t,x}$ is bounded.

This gives a viscosity solution of semilinear parabolic PDE with Neumann boundary condition by Pardoux and Zhang (1998).
Consider the following Markovian BSDEs which is indexed by 
\((t, x) \in [0, T] \times \mathbb{R}^m\) with a random terminal time 
\(\tau := \inf\{s \geq t : X_{s}^{t,x} \notin O\} \wedge T\) for a bounded open set \(O \subset \mathbb{R}^p\).

\[
Y_{s}^{t,x} = h(X_{T}^{t,x}) + \int_{s \land \tau}^{T} g(r, X_{r}^{t,x}, Y_{r}^{t,x}, Z_{r}^{t,x})dr - \int_{s \land \tau}^{T} Z_{r}^{t,x} dW_r
\]

where \(X_{s}^{t,x} = x + \int_{t}^{s} b(r, X_{r}^{t,x})dr + \int_{t}^{s} \sigma(r, X_{r}^{t,x})dW_r\)

**Example**

Assume \(O = (a, b) \subset \mathbb{R}\). Let

\[
X_{s}^{t,x} := x + \sigma(W_s - W_t)
\]

\(\tau := \inf \left\{s \geq t : X_{s}^{t,x} \notin O\right\} \wedge T\)

where \(\sigma\) is a constant. Then, \(X_{\tau}^{t,x}\) is Lipschitz in underlying BM.
This gives a probabilistic representation of a classical solution of semilinear parabolic PDE. (Ladyzenskaja at al. 1968). Peng (1992) used strong version of Ladyzenskaja’s theorem to prove this for Lipschitz driver. We used our result to recover part of original Ladyzenskaja’s theorem.
The regularity of the terminal condition gives the bound of $Z$ (and $Y$) in our BSDE. Then we can apply most of standard BSDE results.

The Lipschitzness is a sufficient, but not necessary condition for bounded Malliavin derivative and this is often easier to check than Malliavin boundedness.

Comparison theorem is crucial in the global existence of solution. In multidimensional superquadratic BSDE, we have solution only for small time if we assume the terminal condition has bounded Malliavin derivative.


Frei, C., Dos Reis, G. (2012). Quadratic FSDE with generalized Burgers’ type nonlinearities, perturbations and large deviations. To appear in: Stochastic and Dynamics


