Google maps and improper Poisson line processes

Oxford Stochastic Analysis Seminar Series

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(joint with David Aldous, work in progress)

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Google
Scale-invariant Random Spatial Networks

Aldous (2012)

- **Input**: set of nodes $x_1, \ldots, x_n$;
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1. **Scale-invariance:**
   \[
   \mathcal{L} (N(\lambda x_1, \ldots, \lambda x_n)) = \mathcal{L} (\lambda N(x_1, \ldots, x_n))
   \]
   for each Euclidean similarity $\lambda$. 
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2. Let $D_1$ be length of fastest route between two points at unit distance apart. We want $\mathbb{E}[D_1] < \infty$. 
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2. Let $D_1$ be length of fastest route between two points at unit distance apart. We want $\mathbb{E}[D_1] < \infty$.

3. Some condition like, consider network derived by connecting all points of unit intensity Poisson point process. Average length per unit area of resulting “fastest route” network should be finite.
Models for SIRSN

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- Hierarchical binary model (randomized direction and location);
- Dynamic proximity graph model;
- Improper Poisson line process.
Antecedents

Frustrated optimization for Roman roads.
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[Map of Roman roads in Britain] [Statue of a Roman emperor]
Improving a network

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Debunks a “natural” statistic for network efficiency. (But see Aldous and Shun 2010.)
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Introducing Line Processes
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compute length of regularizable curve by counting mean number of hits by unit-intensity invariant Poisson line process (Steinhaus).
Random Line Processes (I)

• How to build a random line process on the plane?

Represent (undirected) lines \( \ell \) by points \((r, \theta)\) on a cylinder (actually a punctured projective plane). Here \(-\infty < r < \infty\) while \(0 \leq \theta < \pi\).

Invariant measure is \( \frac{1}{2} dr d\theta \).

Poisson point process on cylinder yields Poisson line process.

Mean number of lines hitting unit segment = 1.
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Random Line Processes (II)

- Variant parametrization:
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Poisson line processes in $\mathbb{R}^d$:

$$\text{Invariant measure now } c \, d\, d\, x \times \nu \, d(\, d\, \varpi).$$

Coordinate $x$ is "twisted" by $\varpi$: measure theory doesn't see this.

Variant parametrization replaces $x$ by $p$, intersection of $\ell$ with reference hyperplane. Invariant measure now $c \, d\sin \theta \, d\, p \times \nu \, d\, p^{-1}(\, d\, \varpi).$
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Improper Poisson line process:

- each line marked with positive speed-limit $v$;
- representing space is now parametrized by $v, r, \theta$ (more generally, in $d$ dimensions, $v, x, \omega$);
- to achieve scale-invariance, invariant measure is $\frac{1}{2} v^{-\gamma} d v d r d \theta$ for positive $\gamma$ (more generally, $c_d v^{-\gamma} d v d r \times \nu_{d-1}(d \omega)$).
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Use lines to go from A to B as fast as legally possible. For which $\gamma$ might we get a decent network?
What is a path? (I)

Seek shortest-time paths (“temporal geodesics” or $\Pi$-geodesics) built using line process $\Pi$. 
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Require $γ > d$, or fast lines will go everywhere.
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- Introduce maximum speed limit, upper-semi-continuous $V : \mathbb{R}^d \to [0, \infty)$. 

Consequence: $\Pi$-geodesics exist if $\Pi$-paths exist.
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- A \( \Pi \)-path is locally Lipschitz, integrates measurable orientation field determined by \( \Pi \), obeys speed limit.
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- Introduce **maximum speed limit**, upper-semi-continuous \(V : \mathbb{R}^d \to [0, \infty)\).
- A \(\Pi\)-path is locally Lipschitz, integrates measurable orientation field determined by \(\Pi\), obeys speed limit.
- If \(\gamma > d\) then:
  - there is an *a priori* random bound on distance travelled by \(\Pi\)-path in fixed time;
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  - paths up to time \(T\), beginning in a compact set, together form a weakly compact set.
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**Consequence:** Π-geodesics exist if Π-paths exist.
What is a path? (II)

Suppose one wishes to connect two points $\xi_1$ and $\xi_2$ in $\mathbb{R}^d$ by a $\Pi$-path. Suppose $\gamma > d$ (turns out to be essential).
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- Construct small balls around $\xi_1$ and $\xi_2$;
- Connect balls by fastest line $\ell$ intersecting both balls;
- Construct daughter nodes on $\ell$ closest to $\xi_1$ and $\xi_2$;
- Recurse.

Borel-Cantelli, et cetera: establish almost sure existence of resulting path. This yields a binary tree representation of the path. Note this is unavoidable if $d > 2!$

A similar but more complicated argument almost surely allows simultaneous construction of paths between all possible pairs $\xi_1$ and $\xi_2$ in $\mathbb{R}^d$.

Exercise: Visualize such paths in case $d = 3$. 
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Simulations (approximate!) of a typical set of routes
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Are \( \Pi \)-geodesics unique? (I)

Suppose now \( d = 2 \) and \( \gamma > 2 \), and we fix \( \xi_1 \) and \( \xi_2 \in \mathbb{R}^2 \). If \( \Pi \) is to generate a network between a finite set of points, then we need to know the \( \Pi \)-geodesic between \( \xi_1 \) and \( \xi_2 \) is almost surely unique.
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Suppose now $d = 2$ and $\gamma > 2$, and we fix $\xi_1$ and $\xi_2 \in \mathbb{R}^2$. If $\Pi$ is to generate a network between a finite set of points, then we need to know the $\Pi$-geodesic between $\xi_1$ and $\xi_2$ is almost surely unique.

- **Theorem**: All non-singleton intersections of $\Pi$-geodesic with lines $\ell$ of $\Pi$ are “line meets line”.
Are Π-geodesics unique? (I)

Suppose now $d = 2$ and $\gamma > 2$, and we fix $\xi_1$ and $\xi_2 \in \mathbb{R}^2$. If Π is to generate a network between a finite set of points, then we need to know the Π-geodesic between $\xi_1$ and $\xi_2$ is almost surely unique.

- **Theorem:** All non-singleton intersections of Π-geodesic with lines $\ell$ of Π are “line meets line”.
  - First, reduce to case of $\ell$ being fastest line in region, with speed $w$. 
Are \(\Pi\)-geodesics unique? (I)

Suppose now \(d = 2\) and \(\gamma > 2\), and we fix \(\xi_1\) and \(\xi_2 \in \mathbb{R}^2\). If \(\Pi\) is to generate a network between a finite set of points, then we need to know the \(\Pi\)-geodesic between \(\xi_1\) and \(\xi_2\) is almost surely unique.

**Theorem:** All non-singleton intersections of \(\Pi\)-geodesic with lines \(\ell\) of \(\Pi\) are “line meets line”.

- First, reduce to case of \(\ell\) being fastest line in region, with speed \(w\).
- Now argue by replacing speed \(v\) by

\[
\text{“cost”} = \frac{\csc \theta}{v} - \frac{\cot \theta}{w}.
\]

where \(\theta\) is angle of line with \(\ell\).
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Suppose now $d = 2$ and $\gamma > 2$, and we fix $\xi_1$ and $\xi_2 \in \mathbb{R}^2$. If $\Pi$ is to generate a network between a finite set of points, then we need to know the $\Pi$-geodesic between $\xi_1$ and $\xi_2$ is almost surely unique.

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\]

where $\theta$ is angle of line with $\ell$.

- Argue that $\Pi$-geodesic hits $\ell$ using line of finite cost.
Are Π-geodesics unique? (II)

So Π-geodesics between $\xi_1$ and $\xi_2$ are made up of countable collection of intervals of lines of $\Pi$. 
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- Fix a given $\ell$ from $\Pi$, and consider the set $S$ of such intervals lying in $\ell$.
- Consider two different finite collections $S_1$ and $S_2$ of $S$, each composed of non-overlapping intervals.

Theorem: given $\xi_1$ and $\xi_2$, almost surely there is just one $\Pi$-geodesic between them.
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  - Consider two different finite collections $S_1$ and $S_2$ of $S$, each composed of non-overlapping intervals.
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Do $\Pi$-geodesics have finite mean length?

Suppose again that $d = 2$ and $\gamma > 2$. 
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- We can upper-bound distance travelled outside a ball by using the “idealized path” construction employed above.
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- The resulting perpetuity can be combined with the “racetrack” bound to establish finite mean length.
Conclusion

The improper line process construction gives a scale-invariant random spatial network for finite sets of points in the plane with $\gamma > 2$. 
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  Example: there is just one singly-infinite geodesic ray from each point.
Questions?


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