Filtration shrinkage, strict local martingales, and the Föllmer measure

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Outline

- Introduction
- The Föllmer measure and a measure extension problem
- Consequences for optional projections
- Path properties
- Examples

Introduction

Filtration shrinkage

- $(\Omega, \mathcal{G}, \mathbb{G}, P)$ filtered probability space, $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$
- $X = (X_t)_{t \geq 0}$ a semimartingale
- $\mathbb{F} \subset \mathbb{G}$ a subfiltration
- ${}^oX = ({}^oX_t, t \geq 0)$, the optional projection of X onto $\mathbb{F}$:

$$E[{}^oX_\tau \mathbf{1}_{\{\tau < \infty\}}] = E[X_\tau \mathbf{1}_{\{\tau < \infty\}}], \text{ all } \mathbb{F}\text{-stopping times } \tau$$

(All filtrations satisfy the usual hypotheses.)

Given properties of X, what can be said about ${}^oX$?

Introduction

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(1) X is a semimartingale for $\mathbb{F}$.

(2) If X is a $\mathbb{G}$-local martingale, then X is an $\mathbb{F}$-local martingale if and only if X is $\mathbb{F}$-special.

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... And many results from filtering theory.

Introduction

Optional projections of martingales are martingales

Suppose X is a $\mathbb{G}$-martingale. Since ${}^oX_t = E[X_t \mid \mathcal{F}_t]$,

$$\begin{aligned} E[{}^oX_t \mid \mathcal{F}_s] &= E[X_t \mid \mathcal{F}_s] \\ &= E[E[X_t \mid \mathcal{G}_s] \mid \mathcal{F}_s] \\ &= E[X_s \mid \mathcal{F}_s] \\ &= {}^oX_s. \end{aligned}$$

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Not true if X is local martingale

- ${}^oX$ may not be a local martingale (e.g. $\mathcal{F}_t \equiv \{\emptyset, \Omega\}$)
- May have FV part with singular paths (wrt $d\langle {}^oX, {}^oX\rangle_t$)

Goal: Understand what causes this behavior.

Introduction

Why? (Besides being an interesting mathematical problem)

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Absence of arbitrage (NFLVR)

$$\Longleftrightarrow$$

Prices (if positive) are **local martingales**
under an equivalent probability measure

- (NFLVR) not stable under information reduction.
- This happens when prices are **strict local martingales**;
- Connections to "bubbles" and "relative arbitrage" (Jarrow, Protter, Shimbo, Heston, Loewenstein, Willard, Karatzas, Fernholz, Ruf, Hugonnier, ...)
- Agents with limited information ($\mathbb{F}$) may perceive arbitrage where there is none (Föllmer-Protter, 2011).

The basic example by Föllmer and Protter (2011)

- $B = (B^1, B^2, B^3)$, 3-dim Brownian motion, $B_0 = (1, 0, 0)$.
- $N = 1/\|B\|$, reciprocal of BES(3) process, a strict local martingale.
- $\mathbb{F}$ generated by B^1.

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Then:

$$E[N_t \mid \mathcal{F}_t] = E\left[\frac{1}{\|B_t\|} \;\middle|\; B_t^1\right] = u(t, B_t^1),$$

where

$$u(t,x) = \sqrt{\frac{2\pi}{t}} \exp\left(\frac{x^2}{2t}\right)\left(1 - \Phi(|x|/\sqrt{t})\right).$$

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One can check that $u(t,x)$ satisfies

$$\frac{1}{2}u_{xx} + u_t = -\frac{1}{t}\delta_0 \quad \text{on} \quad (0,\infty)\times\mathbb{R}.$$

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- But this is highly specific to this particular example, and not very illuminating.
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Remark. Let $\mathbb{F}'$ be generated by (B^1, B^2). Then $E[N_t \mid \mathcal{F}'_t]$ is still a local martingale.

Setup

- $(\Omega, \mathcal{G}, \mathbb{G}, P)$, $\mathcal{G} = \mathcal{G}_\infty$
- $\mathbb{G}$ is right-continuous modification of a *standard system* $\mathbb{G}^0$.
  I.e., $\mathcal{G}_t^0$ are standard Borel, and $\mathbb{G}^0$ satisfies a closure property.
  Important: $\mathbb{G}$ is **not augmented** with the P-nullsets.
- N is a local martingale, P-a.s. positive and càdlàg, $N_0 = 1$.
- $\mathbb{F} \subset \mathbb{G}$ a right-continuous subfiltration
- **Our focus:** $E^P[N_t \mid \mathcal{F}_t]$, especially the FV part.

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Note:

- $P(\tau_0 < \infty) = 0$.
- $\mathcal{G}$ and $\mathcal{G}_{\tau_0-} = \bigvee_{n\geq 1} \mathcal{G}_{\tau_n-}$ coincide up to P-nullsets.

The Föllmer measure

- Define Q_n on $\mathcal{G}_{\tau_n}$ via $\frac{dQ_n}{dP} = N_{\tau_n}$.
- $(Q_n)_{n\geq 1}$ are consistent: $Q_{n+1} = Q_n$ on $\mathcal{G}_{\tau_n -}$
- Extension theorem by Parthasarathy (1967) gives

$$Q_0 \text{ on } \mathcal{G}_{\tau_0 -} \text{ such that } Q_0 = Q_n \text{ on } \mathcal{G}_{\tau_n -}, \text{ all } n.$$

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Note:

- Q_0 is uniquely determined by N.
- $Q_0(\tau_0 > t) = E^P[N_t] < 1$ if N is strict. Hence there is "room" to extend Q_0 to all of $\mathcal{G}$ in a nontrivial way.
- There could be many extensions; However...

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This uses that $\mathbb{G}^0$ is a standard system.

Lemma. Let Q be any extension of Q_0 to $\mathcal{G}$. Then $P \overset{loc}{\ll} Q$, and

$$\frac{dP|_{\mathcal{G}_t}}{dQ|_{\mathcal{G}_t}} = M_t := \frac{1}{N_t}\mathbf{1}_{\{\tau_0 > t\}}.$$

A measure extension problem

Measure extension problem (MEP). Given Q_0, find a probability Q on $(\Omega, \mathcal{G})$ with the following properties:

(i) $Q = Q_0$ on $\mathcal{G}_{\tau_0-}$

(ii) The restrictions $P|_{\mathcal{F}_t}$ and $Q|_{\mathcal{F}_t}$ are equivalent for each $t \geq 0$.

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Remarks.

- If Q solves the MEP, then $E^P[N_t \mid \mathcal{F}_t]$ and $E^Q[M_t \mid \mathcal{F}_t]$ are unique up to P- and Q-evanescence.

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- The MEP is more likely to have a solution if:
  - $\mathbb{F}$ is small,
  - N is not "too strict" (the set $\{\tau_0 < \infty\}$ is small)

Connection to the problem at hand

Lemma. Let Q be any extension of Q_0 to $\mathcal{G}$, and fix $t \geq 0$. The following are equivalent:

(i) The restrictions $P|_{\mathcal{F}_t}$ and $Q|_{\mathcal{F}_t}$ are equivalent,

(ii) $E^Q[M_t \mid \mathcal{F}_t] > 0$, Q-a.s.,

(iii) $Q(\tau_0 > t \mid \mathcal{F}_t) > 0$, Q-a.s.

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If either condition holds, then

$$Q(\tau_0 > t \mid \mathcal{F}_t) = E^Q[M_t \mid \mathcal{F}_t] E^P[N_t \mid \mathcal{F}_t], \quad P\text{- and } Q\text{-a.s.}$$

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Consequence: Can study $E^Q[M_t \mid \mathcal{F}_t]$ and $Q(\tau_0 > t \mid \mathcal{F}_t)$. This is a simplification: the former is a martingale, the latter only involves a single stopping time.

The structure of the optional projection

Multiplicative Doob-Meyer decomposition:

$$Q(\tau_0 > t \mid \mathcal{F}_t) = e^{-\Lambda_t} K_t, \tag{1}$$

where Λ is nondecreasing $\mathbb{F}$-predictable with $\Lambda_0 = 0$,
K is $(\mathbb{F}, Q)$-local martingale K with $K_0 = 1$.

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Theorem. Suppose Q solves the MEP. Then $E^P[N_t \mid \mathcal{F}_t]$ is an $(\mathbb{F}, P)$-supermartingale, with multiplicative decomposition

$$E^P[N_t \mid \mathcal{F}_t] = e^{-\Lambda_t} U_t,$$

with Λ as in (1), and U an $(\mathbb{F}, P)$-local martingale. U is a true martingale provided K in (1) is a true $(\mathbb{F}, Q)$-martingale.

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Corollary. Suppose N is a strict $(\mathbb{G}, P)$-local martingale. If $E^P[N_t \mid \mathcal{F}_t]$ is again an $(\mathbb{F}, P)$-local martingale, then the MEP has no solution.

Consequence: The MEP cannot always be solved: The projection of $N = 1/\|B\|$ onto $\mathbb{F}'$ generated by (B^1, B^2) is a local martingale.

The structure of the optional projection

Using the Jeulin-Yor Theorem from the theory of filtration expansion, we get:

Theorem. Let $\mathbb{F}^{\tau_0}$ be the progressive expansion of $\mathbb{F}$ with τ_0: the smallest filtration that contains $\mathbb{F}$, satisfies the usual hypotheses (with respect to Q), and makes τ_0 a stopping time. Suppose Q solves the MEP. Then

$$\mathbf{1}_{\{\tau_0 \leq t\}} - \int_0^{t \wedge \tau_0} d\Lambda_s$$

is a $(\mathbb{F}^{\tau_0}, Q)$-uniformly integrable martingale, where Λ is as in (1).

In other words: The FV part of (the multiplicative decomposition of) $E^P[N_t \mid \mathcal{F}_t]$ can be interpreted as the predictable compensator (under Q) of the explosion time τ_0.

The structure of the optional projection

This is a bit counterintuitive:

- τ_0 is invisible from the point of view of P (it never occurs).
- But in the smaller filtration its presence becomes noticeable via the drift of $E^P[N_t \mid \mathcal{F}_t]$.
- Reason: when restricted to $\mathbb{F}$, P becomes equivalent to a measure Q under which τ_0 *does* occur.

Path properties in a “topological” setting

Specialize the setup:

- E is a locally compact topological space with countable base, $E_\Delta = E \cup \{\Delta\}$, some $\Delta \notin E$.
- Ω is all right-continuous $\omega : \mathbb{R}_+ \to E_\Delta$ absorbed at Δ, with left limits on $(0, \zeta(\omega))$, where $\zeta(\omega) = \inf\{t : \omega(t) = \Delta\}$.
- $Y_t(\omega) = \omega(t)$ is the coordinate process.
- $\mathcal{G}_t = \cap_{u>t} \sigma(Y_s : s \leq u)$
- $N_t = \dfrac{1}{h(Y_t)} \mathbf{1}_{\{\tau_0 > t\}}$, some $h : E_\Delta \to [0, \infty)$, continuous on E.

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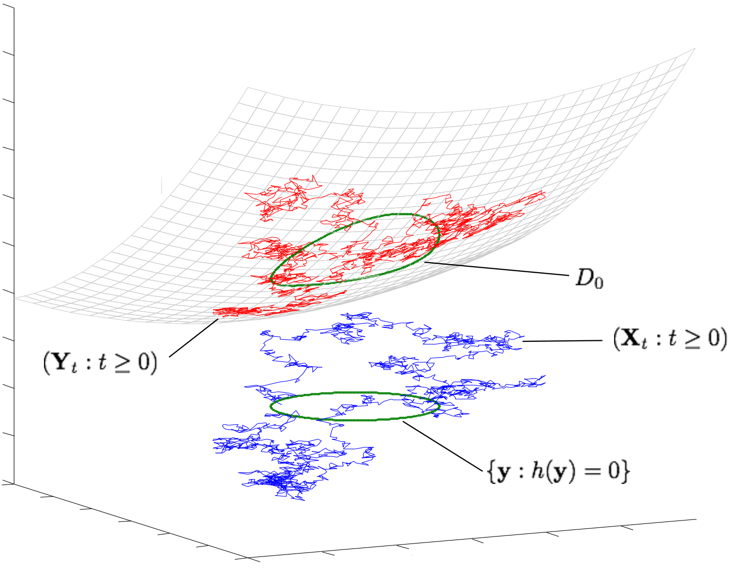
The subfiltration:

- D is a metrizable topological space, $D_\Delta = D \cup \{\Delta\}$.
- $\pi : E \to D$ continuous, set $\pi(\Delta) = \Delta$.
- $X_t = \pi(Y_t)$, $t \geq 0$.
- $\mathcal{F}_t = \cap_{u>t} \sigma(X_s : s \leq u)$.

Path properties in a “topological” setting



Path properties in a "topological" setting

Theorem. Assume Q solves the MEP, and let Λ be as in (1). Then the random measure $d\Lambda_t$ is supported on the set $\{t : X_{t-} \in \overline{D_0}\}$, where $\overline{D_0}$ is the closure in D of

$$\begin{aligned} D_0 &= \pi \circ h^{-1}(\{0\}) \\ &= \{x \in D : x = \pi(y) \text{ for some } y \in E \text{ with } h(y) = 0\}\,. \end{aligned}$$

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Corollary. Assume D is a subset of $\mathbb{R}^k$ for some k, and that the law of X_t admits a density for almost every $t > 0$. Then, if $\overline{D_0}$ is a nullset in $\mathbb{R}^k$, the paths of Λ are singular.

Solving the MEP: Brownian motion

Specialize further:

- $E = \mathbb{R}^q$, some $q \in \mathbb{N}$.
- P is Wiener measure, so that Y is q-dimensional Brownian motion with $Y_0 = y_0 \in \mathbb{R}^q$.
- h is such that $h(y_0) = 1$ and $\frac{1}{h}$ is harmonic on $\mathbb{R}^q \setminus E_0$, where
$$E_0 = h^{-1}(\{0\}).$$
- $\pi : E \to E$ is linear, $D = \pi(\mathbb{R}^q)$. We set $p = \dim D = \mathrm{rank}\ \pi$.

Notation: $|\cdot|$ is the Euclidean norm, ∇ is the gradient.

Solving the MEP: Brownian motion

Theorem. Define functions

$$F(t) = E^P\left[\frac{|\nabla \log h(Y_t)|}{h(Y_t)}\right],$$

$$G(t,x) = E^P\left[\frac{|\pi(\nabla \log h(Y_t))|}{h(Y_t)} \;\middle|\; \pi(Y_t) = x\right].$$

Assume h is such that F is locally bounded on $[0,\infty)$, and G is locally bounded on $(0,\infty)\times D$. Then the MEP has a solution Q, given as the measure on path space induced by the SDE

$$Y_t = y_0 + W_t - \int_0^{t\wedge\tau_0} \nabla \log h(Y_s)ds,$$

where W is q-dim. Brownian motion.

Solving the MEP

Strategy of proof:

(1) Show $Y_t = y_0 + W_t - \int_0^{t \wedge \tau_0} \nabla \log h(Y_s) ds$ has a weak solution. Let Q be its law, and show that Q extends Q_0.

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(2) Describe the law of $X = \pi(Y)$ under P and Q.

(3) Use this to show the law of $(X_s : s \leq t)$ under P and Q are equivalent for each t. Since X generates $\mathbb{F}$, this implies

$$P|_{\mathcal{F}_t} \sim Q|_{\mathcal{F}_t}, \qquad t \geq 0.$$

Solving the MEP

Lemma. We have

$$X_t = X_0 + B_t^P,$$

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Lemma. The process X can be decomposed as

$$X_t = X_0 + B_t^Q + \int_0^t \theta_s ds \quad \text{for all} \quad t \geq 0, \qquad Q\text{-a.s.},$$

with B^Q the image under $\pi|_D$ of p-dim $(\mathbb{F}, Q)$-BM, and

$$\int_0^t E^Q \left[|\theta_s| \right] ds < \infty, \quad t \geq 0,$$

$$\theta_t = E^Q \left[\pi(\nabla \log h(Y_t)) \mathbf{1}_{\{\tau_0 > t\}} \mid \mathcal{F}_t \right] \quad Q\text{-a.s.}, \quad t \geq 0.$$

Solving the MEP

Key lemma:

Lemma. For each $t \geq 0$, we have

$$\int_0^t |\theta_s|^2 ds < \infty \qquad Q\text{-a.s.}$$

This gives local equivalence.

Solving the MEP

- By Lemma, $Z_t = \exp\left(\int_0^t \theta_s^\top dB_s^Q - \frac{1}{2}\int_0^t |\theta_s|^2 ds\right)$ is a strictly positive $(\mathbb{F}, Q)$-local martingale.

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- Let R be the Föllmer measure on $\mathcal{F}_{\rho_0 -}$ where

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- $P|_{\mathcal{F}_t} \ll Q|_{\mathcal{F}_t} \ll R|_{\mathcal{F}_t} = P|_{\mathcal{F}_t}$.

Examples

The example by Föllmer and Protter

- $E = \mathbb{R}^3$,
- $y_0 = (1, 0, 0)$,
- $h(y) = |y|$, then $1/h$ is harmonic on $\mathbb{R}^3 \setminus \{0\}$
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Conditions of theorem (local boundedness by direct calculation):

$$F(t) = E^P[N_t^2]$$

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Singular FV part of the projection:

$$D_0 = \pi \circ h^{-1}(\{0\}) = \{0\} \subset \mathbb{R}.$$

Hence $d\Lambda_t$ is supported on $\{t : Y_t^1 = 0\}$.

Examples

Noisy observation of two components

- $E = \mathbb{R}^4$,
- $y_0 = (1, 0, 0, 0)$,
- $h(y) = |(y_1, y_2, y_3, 0)|$, $1/h$ harmonic outside the y_4-axis
- $\pi(y) = Ay$, where for some $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$,

$$A = \begin{pmatrix} 1 & 0 & 0 & \alpha_1 \\ 0 & 1 & 0 & \alpha_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

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Note: If $\alpha_1 = \alpha_2 = 0$, the MEP has no solution.

Examples

New examples from old

- Consider $h_1, \ldots, h_m$ with $\dfrac{1}{h_i}$ harmonic outside $h_i^{-1}(\{0\})$.
- Let $E_0 = \bigcup_{i=1}^m h_i^{-1}(\{0\})$, and define h so that

$$\frac{1}{h} = \frac{1}{h_1} + \cdots + \frac{1}{h_m} \quad \text{on} \quad E \setminus E_0.$$

- Then $1/h$ is harmonic outside E_0 and we have

$$\frac{1}{h}\nabla \ln h = \frac{1}{h_1}\nabla \ln h_1 + \cdots + \frac{1}{h_m}\nabla \ln h_m.$$

Thus: if all h_i satisfy the conditions of the theorem, h does also.

Examples

Conditioned martingales:

- Let Q be Wiener measure, Y is p-dimensional $(\mathbb{G}, Q)$-BM.
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- Then

$$N_t = \frac{1}{M_t}\mathbf{1}_{\{\tau_0 > t\}}$$

  is a $(\mathbb{G}, P)$-local martingale, strict iff $Q(\tau_0 < \infty) > 0$.

Examples

Conditioned martingales:

- Let $\pi : \mathbb{R}^p \to \mathbb{R}^q$ be continuous,
- Set $X_t = \pi(Y_t)$,
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Given the P and Q above, we have $Q_0 = Q|_{\mathcal{G}_{\tau_0 -}}$.

Hence Q solves the MEP if and only if

$$Q(\tau_0 > t \mid \mathcal{F}_t) > 0, \quad Q\text{-a.s.}, \quad t \geq 0.$$

This is easy to check in many cases.

Examples

Conditioned martingales: BES(3), again.

- $Y_t = (Y_t^1, Y_t^2)$,
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Note: Not easy to characterize the FV part of $E^P[N_t \mid \mathcal{F}_t]$. This requires the multiplicative decomposition

$$Q(\tau_0 > t \mid \mathcal{F}_t) = e^{-\Lambda_t} K_t.$$

Summary

- Optional projections of local martingales need not be local martingales.
- They acquire drift, which may be singular.
- This drift can be understood as the predictable compensator of an stopping time τ_0.
- τ_0 is the explosion time under some Q with $P \overset{loc}{\ll} Q$, of the local martingale.
- Q is solves an extension problem based on the Föllmer measure.
- Have conditions for existence of solution in some cases.
- Further research:
  - Generalize these conditions
  - Intermediate case: MEP has no solution, but projection is still local martingale.