Dynamic Econometric Modeling of Volatility, Correlation and Connectedness in Financial Markets

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Prologue: Reading


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Prologue

- Throughout: Desirability of *conditional* risk measurement
- Aggregation level
  - Portfolio-level (aggregated) Risk measurement
  - Asset-level (disaggregated): Risk management
- Frequency of data observations
  - Low-frequency vs. high-frequency data
  - Parametric vs. nonparametric volatility measurement
- Object measured and modeled
  - Conditional variances, intervals, densities
- Dimensionality reduction in “vast” multivariate environments
  - Factor structure
- Beyond correlation: Network connectedness and systemic risk
What’s in the Data?
Returns

Figure: Time Series of Daily NYSE Returns.
Key Fact 1: Returns are Approximately Serially Uncorrelated

Figure: Correlogram of Daily NYSE Returns.
Key Fact 2: Returns are not Gaussian

Figure: Histogram and Statistics for Daily NYSE Returns.
Key Fact 3: Returns are Conditionally Heteroskedastic

Figure: Time Series of Daily Squared NYSE Returns.
Key Fact 3: Returns are Conditionally Heteroskedastic II

Figure: Correlogram of Daily Squared NYSE Returns.
Why Care About Volatility Dynamics?
Everything Changes when Volatility is Dynamic

- Risk management
- Portfolio allocation
- Asset pricing
- Hedging
- Trading
Risk Management

Individual asset returns:

\[ r \sim (\mu, \Sigma) \]

Portfolio returns:

\[ r_p = \lambda' r \sim (\lambda' \mu, \lambda' \Sigma \lambda) \]

If \( \Sigma \) varies, we need to track time-varying portfolio risk, \( \lambda' \Sigma_t \lambda \)
Optimal portfolio shares $w^*$ solve:

$$\min_w \ w' \Sigma w$$

s.t. $w' \mu = \mu_p$,

Importantly, $w^* = f(\Sigma)$

If $\Sigma$ varies, we have $w_t^* = f(\Sigma_t)$
Asset Pricing I: Sharpe Ratios

Standard Sharpe:

\[
\frac{E(r_{it} - r_{ft})}{\sigma}
\]

Conditional Sharpe:

\[
\frac{E(r_{it} - r_{ft})}{\sigma_t}
\]
Asset Pricing II: CAPM

Standard CAPM:

\[(r_{it} - r_{ft}) = \alpha + \beta(r_{mt} - r_{ft})\]

\[\beta = \frac{\text{cov}((r_{it} - r_{ft}), (r_{mt} - r_{ft}))}{\text{var}(r_{mt} - r_{ft})}\]

Conditional CAPM:

\[\beta_t = \frac{\text{cov}_t((r_{it} - r_{ft}), (r_{mt} - r_{ft}))}{\text{var}_t(r_{mt} - r_{ft})}\]
Black-Scholes:

\[ C = N(d_1)S - N(d_2)Ke^{-r\tau} \]

\[
\begin{align*}
  d_1 &= \frac{\ln(S/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \\
  d_2 &= \frac{\ln(S/K) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}
\end{align*}
\]

\[ P_C = BS(\sigma, ...) \]

(Standard Black-Scholes options pricing)

Completed different when \( \sigma \) varies!
Hedging

- Standard delta hedging

\[ \Delta H_t = \delta \Delta S_t + u_t \]

\[ \delta = \frac{\text{cov}(\Delta H_t, \Delta S_t)}{\text{var}(\Delta S_t)} \]

- Dynamic hedging

\[ \Delta H_t = \delta_t \Delta S_t + u_t \]

\[ \delta_t = \frac{\text{cov}_t(\Delta H_t, \Delta S_t)}{\text{var}_t(\Delta S_t)} \]
Trading

- Standard case: no way to trade on fixed volatility

- Time-varying volatility I: Options (indirect) Take position according to whether: \( P_C > f(\sigma_{t+h,t}, \ldots) \), \( P_C < f(\sigma_{t+h,t}, \ldots) \)

- Time-varying volatility II: Volatility swaps (direct) Effectively futures contracts written on underlying “realized volatility”
Some Warm-Up
Unconditional Volatility Measures

Variance: $\sigma^2 = E(r_t - \mu)^2$ (or standard deviation: $\sigma$)

Mean Absolute Deviation: $MAD = E|r_t - \mu|$

Interquartile Range: $IQR = 75\% - 25\%$

$p\%$ Value at Risk $(VaR^p))$: $x$ s.t. $P(r_t < x) = p$

Outlier probability: $P|r_t - \mu| > 5\sigma$

Tail index: $\gamma$ s.t. $P(r_t > r) = k r^{-\gamma}$

Kurtosis: $K = E(r - \mu)^4/\sigma^4$
Dangers of HS-VaR, Take I

Figure: Cumulative S&P500 Loss (left-scale, dashed) and 1% 10-day HS-VaR (right scale, solid), July 1, 2008 - December 31, 2009. The dashed line shows the cumulative percentage loss on an S&P500 portfolio from July 2008 through December 2009. The solid line shows the daily 10-day 1% HS-VaR based on a 500-day moving window of historical returns.
Dangers of HS-VaR, Take II

Figure: True Exceedance Probabilities of Nominal 1% HS-VaR When Volatility is Persistent. We simulate returns from a realistically-calibrated dynamic volatility model, after which we compute 1-day 1% HS-VaR using a rolling window of 500 observations. We plot the daily series of true conditional exceedance probabilities, which we infer from the model. For visual reference we include a horizontal line at the desired 1% probability level.
Dangers of HS-VaR, Take III

The unconditional HS-VaR perspective encourages incorrect rules of thumb, like scaling by $\sqrt{h}$ to convert 1-day VaR into h-day VaR.
Conditional VaR is Generally More Appropriate

Conditional VaR \( (\text{VaR}^p_{T+1|T}) \) solves:

\[
p = \Pr_T(r_{T+1} \leq -\text{VaR}_{T+1|T}^p) = \int_{-\infty}^{-\text{VaR}_{T+1|T}^p} f_T(r_{T+1}) \, dr_{T+1}
\]

(\( f_T(r_{T+1}) \) is density of \( r_{T+1} \) conditional on time- \( T \) information)
But VaR of any Flavor has Issues

- VaR is silent regarding expected loss when VaR is exceeded (fails to assess the entire distributional tail)
- VaR fails to capture beneficial effects of portfolio diversification

Expected shortfall:

\[ ES_T^{p} = p^{-1} \int_{0}^{p} VaR_T^{\gamma} d\gamma \]

- ES assesses the entire distributional tail
- ES captures the beneficial effects of portfolio diversification
Exponential Smoothing and RiskMetrics

\[
\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1 - \lambda) r_{t-1}^2
\]

\[
\sigma_t^2 = \sum_{j=0}^{\infty} \varphi_j r_{t-1-j}^2
\]

\[
\varphi_j = (1 - \lambda) \lambda^j
\]

- Many initializations possible (\(r_1^2\), sample variance, etc.)

\[
RM-VaR_p^{T+1|T} \equiv \sigma_{T+1} \Phi_p^{-1}
\]

(Assumes a particular volatility dynamic and conditional normality)
Conditional variance for the $k$-day return in RM is

$$\text{Var}(r_{t+k} + r_{t+k-1} + \ldots + r_{t+1} \mid F_t) \equiv \sigma^2_{t:t+k \mid t} = k \sigma^2_{t+1}$$

- Random walk for variance
- Random walk plus noise model for squared returns
- Optimal volatility forecast at any horizon is simply the current smoothed value
- Flat volatility term structure is not realistic!
Rigorous Modeling I

Conditional Portfolio-Level Volatility Dynamics from “Daily” Data
Conditional Return Distributions

\[ f(r_t) \text{ vs. } f(r_t | \Omega_{t-1}) \]

Key 1: \( E(r_t | \Omega_{t-1}) \)

Are returns conditional mean independent? Arguably yes.

Returns are (arguably) approximately serially uncorrelated, and (arguably) approximately free of additional non-linear conditional mean dependence.
Key 2: \( \text{var}(r_t | \Omega_{t-1}) = E((r_t - \mu)^2 | \Omega_{t-1}) \)

Are returns conditional variance independent? No way!

Squared returns serially correlated, often with very slow decay.
The Standard Model
(Linearity Indeterministic Process with iid Innovations)

\[ y_t = \sum_{i=0}^{\infty} b_i \varepsilon_{t-i} \]

\[ \varepsilon \sim iid \ (0, \sigma_{\varepsilon}^2) \quad \sum_{i=0}^{\infty} b_i^2 < \infty \quad b_0 = 1 \]

Uncond. mean: \( E(y_t) = 0 \) (constant)

Uncond. variance: \( E(y_t - E(y_t))^2 = \sigma_{\varepsilon}^2 \sum_{i=0}^{\infty} b_i^2 \) (constant)

Cond. mean: \( E(y_t \mid \Omega_{t-1}) = \sum_{i=1}^{\infty} b_i \varepsilon_{t-i} \) (varies)

Cond. variance: \( E([y_t - E(y_t \mid \Omega_{t-1})]^2 \mid \Omega_{t-1}) = \sigma_{\varepsilon}^2 \) (constant)
The Standard Model, Continued

k-Step-Ahead Least Squares Forecasting

\[ E(y_{t+k} \mid \Omega_t) = \sum_{i=0}^{\infty} b_{k+i} \varepsilon_{t-i} \]

Associated prediction error:

\[ y_{t+k} - E(y_{t+k} \mid \Omega_t) = \sum_{i=0}^{k-1} b_i \varepsilon_{t+k-i} \]

Conditional prediction error variance:

\[ E([y_{t+k} - E(y_{t+k} \mid \Omega_t)]^2 \mid \Omega_t) = \sigma_{\varepsilon}^2 \sum_{i=0}^{k-1} b_i^2 \]

Key: Depends only on k, not on \( \Omega_t \)
ARCH(1) Process

\[ r_t \mid \Omega_{t-1} \sim N(0, h_t) \]

\[ h_t = \omega + \alpha r_{t-1}^2 \]

\[ E(r_t) = 0 \]

\[ E(r_t - E(r_t))^2 = \omega/(1 - \alpha) \]

\[ E(r_t \mid \Omega_{t-1}) = 0 \]

\[ E([r_t - E(r_t \mid \Omega_{t-1})]^2 \mid \Omega_{t-1}) = \omega + \alpha r_{t-1}^2 \]
GARCH(1,1) Process

“Generalized ARCH”

\[ r_t \mid \Omega_{t-1} \sim N(0, h_t) \]

\[ h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1} \]

\[ E(r_t) = 0 \]

\[ E(r_t - E(r_t))^2 = \omega/(1 - \alpha - \beta) \]

\[ E(r_t \mid \Omega_{t-1}) = 0 \]

\[ E([r_t - E(r_t \mid \Omega_{t-1})]^2 \mid \Omega_{t-1}) = \omega + \alpha r_{t-1}^2 + \beta h_{t-1} \]
GARCH-Based 1-Day VaR

\[ \text{GARCH-VaR}^p_{T+1|T} \equiv \sigma_{T+1} \Phi_p^{-1} \]

- Assumes Gaussian conditional distribution
- Consistent with fat tails of unconditional return distributions
- Can be extended to allow for fat-tailed conditional distributions
Figure: Cumulative S&P500 Loss (dots, left scale) and 1% 10-day RM-VaR and GARCH-VaR (solid and dashed, right scale), July 1, 2008 - December 31, 2009.
Unified Theoretical Framework

- Volatility dynamics (of course, by construction)
- Conditional symmetry translates into unconditional symmetry
- Volatility clustering produces unconditional leptokurtosis
Tractable Empirical Framework

\[ L(\theta; r_1, \ldots, r_T) \approx f(r_T|\Omega_{T-1}; \theta) f(r_{T-1}|\Omega_{T-2}; \theta) \ldots f(r_{p+1}|\Omega_p; \theta) \]

If the conditional densities are Gaussian,

\[ f(r_t|\Omega_{t-1}; \theta) = \frac{1}{\sqrt{2\pi}} h(t(\theta))^{-1/2} \exp \left( -\frac{1}{2} \frac{r_t^2}{h(t(\theta))} \right) \]

\[ \ln L(\theta; r_{p+1}, \ldots, r_T) \approx -\frac{T - p}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=p+1}^{T} \ln h(t(\theta)) - \frac{1}{2} \sum_{t=p+1}^{T} \frac{r_t^2}{h(t(\theta))} \]
The Squared Return as a Noisy Volatility Proxy

Note that we can write:

\[ r_t^2 = h_t + \nu_t \]

Thus \( r_t^2 \) is a noisy indicator of \( h_t \)

Various approaches handle the noise in various ways.
GARCH(1,1) and Exponential Smoothing

Exponential smoothing recursion:

\[ \bar{r}_t^2 = \gamma r_t^2 + (1 - \gamma) \bar{r}_{t-1}^2 \]

Back substitution yields:

\[ \bar{r}_t^2 = \sum w_j r_{t-j}^2 \]

where \( w_j = \gamma (1 - \gamma)^j \)

But in GARCH(1,1) we have:

\[ h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1} \]

\[ h_t = \frac{\omega}{1 - \beta} + \alpha \sum \beta^{j-1} r_{t-j}^2 \]
Fat-Tailed Conditional Densities: t-GARCH

If \( r \) is conditionally Gaussian, then \( \frac{r_t}{\sqrt{h_t}} \sim N(0, 1) \)

But often with high-frequency data, \( \frac{r_t}{\sqrt{h_t}} \sim \text{fat tailed} \)

So take:

\[
r_t = h_t^{1/2} z_t
\]

\[
\text{iid } \quad z_t \sim \frac{t_d}{\text{std}(t_d)}
\]
Univariate GARCH(1,1) Variance Targeting

Sample unconditional variance:

\[ \hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^{T} r_t^2 \]

Implied unconditional GARCH(1,1) variance:

\[ \sigma^2 = \frac{\omega}{1 - \alpha - \beta} \]

We can constrain \( \sigma^2 = \hat{\sigma}^2 \) by constraining:

\[ \omega = (1 - \alpha - \beta)\hat{\sigma}^2 \]

– Saves a degree of freedom and ensures reasonableness
Additional GARCH material starts here...
Conditional variance is a serially correlated random variable

\[ h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1} \]

Prediction error variance depends on \( \Omega_{t-1} \)
e.g., 1-step-ahead prediction error variance is now

\[ h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1} \]
ARMA Representation in Squares

\[ r_t^2 \text{ has the ARMA}(1,1) \text{ representation:} \]

\[ r_t^2 = \omega + (\alpha + \beta)r_{t-1}^2 - \beta \nu_{t-1} + \nu_t, \]

where \( \nu_t = r_t^2 - h_t \).
Variations on the GARCH Theme

- Regression with GARCH Disturbances
- Incorporating Exogenous Variables
- Asymmetric Response and the Leverage Effect:
- Fat-Tailed Conditional Densities
- Time-Varying Risk Premia
Regression with GARCH Disturbances

\[ y_t = x'_t \beta + \varepsilon_t \]

\[ \varepsilon_t | \Omega_{t-1} \sim N(0, h_t) \]
h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1} + \gamma' z_t

\gamma is a parameter vector

z is a set of positive exogenous variables.
Asymmetric Response and the Leverage Effect I: TARCH

Standard GARCH: \( h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1} \)

TARCH: \( h_t = \omega + \alpha r_{t-1}^2 + \gamma r_{t-1}^2 D_{t-1} + \beta h_{t-1} \)

\[
D_t = \begin{cases} 
1 & \text{if } r_t < 0 \\
0 & \text{otherwise} 
\end{cases}
\]

positive return (good news): \( \alpha \) effect on volatility

negative return (bad news): \( \alpha + \gamma \) effect on volatility

\( \gamma \neq 0 \): Asymmetric news response

\( \gamma > 0 \): “Leverage effect”
Asymmetric Response II: E-GARCH

\[
\ln(h_t) = \omega + \alpha \frac{\left| r_{t-1} \right|}{h_{t-1}^{1/2}} + \gamma \frac{r_{t-1}}{h_{t-1}^{1/2}} + \beta \ln(h_{t-1})
\]

- Log specification ensures that the conditional variance is positive.
- Volatility driven by both size and sign of shocks
- Leverage effect when \( \gamma < 0 \)
Time-Varying Risk Premia: GARCH-M

Standard GARCH regression model:

\[ y_t = x_t' \beta + \varepsilon_t \]

\[ \varepsilon_t | \Omega_{t-1} \sim N(0, h_t) \]

GARCH-M model is a special case:

\[ y_t = x_t' \beta + \gamma h_t + \varepsilon_t \]

\[ \varepsilon_t | \Omega_{t-1} \sim N(0, h_t) \]
A GARCH(1,1) Example

Figure: GARCH(1,1) Estimation, Daily NYSE Returns.
A GARCH(1,1) Example

**Figure**: Correlogram of Squared Standardized GARCH(1,1) Residuals, Daily NYSE Returns.
A GARCH(1,1) Example

**Figure:** Estimated Conditional Standard Deviation, Daily NYSE Returns.
A GARCH(1,1) Example

Figure: Conditional Standard Deviation, History and Forecast, Daily NYSE Returns.
After Exploring Lots of Possible Extensions...

Dependent Variable: R
Method: ML - ARCH (Marquardt) - Student's t distribution
Date: 04/10/12   Time: 13:48
Sample (adjusted): 2 3461
Included observations: 3460 after adjustments
Convergence achieved after 19 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(4) + C(5)*RESID(-1)^2 + C(6)*RESID(-1)^2*(RESID(-1)<0)
+ C(7)*GARCH(-1)

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<th>Coefficient</th>
<th>Std. Error</th>
<th>z-Statistic</th>
<th>Prob.</th>
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<td>@SQRT(GARCH)</td>
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Variance Equation

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<tr>
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Additional GARCH material ends here…
Rigorous Modeling II

Conditional Portfolio-Level Volatility Dynamics from High-Frequency Data
Intraday Data and Realized Volatility

\[ dp(t) = \mu(t)dt + \sigma(t)dW(t) \]

\[ RV_t(\Delta) \equiv \sum_{j=1}^{N(\Delta)} \left( p_{t-1+j\Delta} - p_{t-1+(j-1)\Delta} \right)^2 \]

\[ RV_t(\Delta) \rightarrow IV_t = \int_{t-1}^{t} \sigma^2(\tau) \, d\tau \]
Microstructure Noise

– State space signal extraction
– AvgRV
– Realized kernel
– Many others

– Interestingly, MSN is rapidly becoming less important as spreads narrow (but they may widen again in crises)
RV is Persistent

**Figure:** S&P500 Daily Returns and Volatilities (Percent). The top panel shows daily S&P500 returns, and the bottom panel shows daily S&P500 realized volatility. We compute realized volatility as the square root of $\text{AvgRV}$, where $\text{AvgRV}$ is the average of five daily RVs each computed from 5-minute squared returns on a 1-minute grid of S&P500 futures prices.
RV is Log-Normal

Figure: S&P500: QQ Plots for Realized Volatility and Log Realized Volatility. The top panel plots the quantiles of daily realized volatility against the corresponding normal quantiles. The bottom panel plots the quantiles of the natural logarithm of daily realized volatility against the corresponding normal quantiles. We compute realized volatility as the square root of $\text{AvgRV}$, where $\text{AvgRV}$ is the average of five daily RVs each computed from 5-minute squared returns on a 1-minute grid of S&P500 futures prices.
Crucially: RV is Long-Memory

Figure: S&P500: Sample Autocorrelations of Daily Realized Variance and Daily Return. The top panel shows realized variance autocorrelations, and the bottom panel shows return autocorrelations, for displacements from 1 through 250 days. Horizontal lines denote 95% Bartlett bands. Realized variance is \( \text{AvgRV} \), the average of five daily RVs each computed from 5-minute squared returns on a 1-minute grid of S&P500 futures prices.
Exact and Approximate Long Memory

Exact long memory:

\[(1 - L)^d RV_t = \beta_0 + \nu_t\]

Corsi (2009) approximation (HAR):

\[RV_t = \beta_0 + \beta_1 RV_{t-1} + \beta_2 RV_{t-5:t-1} + \beta_3 RV_{t-21:t-1} + \nu_t\]

Even better:

\[\log RV_t = \beta_0 + \beta_1 \log RV_{t-1} + \beta_2 \log RV_{t-5:t-1} + \beta_3 \log RV_{t-21:t-1} + \nu_t\]

– Ensures positivity and promotes normality
RV - VaR^p_{T+1|T} = \hat{RV}_{T+1|T} \Phi_p^{-1},

where \( \Phi_p^{-1} \) is the relevant standard normal quantile

Easily adapted for multi-period variance forecasts:

\[
RV_{t:t+9} = \beta_0 + \beta_1 RV_{t-1} + \beta_2 RV_{t-5:t-1} + \beta_3 RV_{t-21:t-1} + \nu_{t:t+9}
\]

\[
RV - VaR^p_{T+10|T} = \hat{RV}_{T+1:T+10|T} \Phi_p^{-1}
\]
Figure: 10-day 1% HAR-VaR and GARCH-VaR, July 1, 2008 - December 31, 2009. The dashed line shows 10-day 1% HAR-VaR based on the HAR forecasting model for 10-day realized volatility. The solid line shows 10-day 1% GARCH-VaR. When computing VaR the 10-day returns divided by the expected volatility are assumed to be normally distributed.
GARCH-RV

\[ \sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \gamma RV_{t-1} \]

- Fine for 1-step

- Multi-step requires “closing the system” with an RV equation
  - Hansen, Huang and Shek (2010), “Realized GARCH”
  - Shephard and Sheppard (2010), “HEAVY”
Separating Jumps

\[ QV_t = IV_t + JV_t \]

where

\[ JV_t = \sum_{j=1}^{J_t} J_{t,j}^2 \]

e.g., we might want to explore:

\[ RV_t = \beta_0 + \beta_1 IV_{t-1} + \beta_2 IV_{t-5:t-1} + \beta_3 IV_{t-21:t-1} + \alpha_1 JV_{t-1} + \alpha_2 JV_{t-5:t-1} + \alpha_3 JV_{t-21:t-1} + \nu_t \]
But How to Separate Jumps?

- Mancini (2001) Truncation:
  \[ TV_t(\Delta) = \sum_{j=1}^{N(\Delta)} \Delta p_{t-1+j\Delta}^2 I(\Delta p_{t-1+j\Delta} < T) \]

- Barndorff-Nielsen and Shephard (2004) Bi-Power:
  \[ BPV_t(\Delta) = \frac{\pi}{2} \frac{N(\Delta)}{N(\Delta) - 1} \sum_{j=1}^{N(\Delta) - 1} |\Delta p_{t-1+j\Delta}| |\Delta p_{t-1+(j+1)\Delta}| \]

- Andersen, Dobrev and Schaumburg (2010) Minimum:
  \[ MinRV_t(\Delta) = \frac{\pi}{\pi - 2} \left( \frac{N(\Delta)}{N(\Delta) - 1} \right)^{N(\Delta) - 1} \sum_{j=1}^{N(\Delta) - 1} \min \{ |\Delta p_{t-1+j\Delta}|, |\Delta p_{t-1+(j+1)\Delta}| \}^2 \]
Modeling Entire Return Distributions: Returns are not Unconditionally Gaussian

Figure: QQ Plot of S&P500 Returns. We show quantiles of daily S&P500 returns from January 2, 1990 to December 31, 2010, against the corresponding quantiles from a standard normal distribution.
Modeling Entire Return Distributions: Returns are Often not Conditionally Gaussian

Figure: QQ Plot of S&P500 Returns Standardized by NGARCH Volatilities. We show quantiles of daily S&P500 returns standardized by the dynamic volatility from a NGARCH model against the corresponding quantiles of a standard normal distribution. The sample period is January 2, 1990 through December 31, 2010. The units on each axis are standard deviations.
Modeling Entire Return Distributions: Issues

- Gaussian QQ plots effectively show calibration of Gaussian VaR at different levels
- Gaussian unconditional VaR is terrible
- Gaussian conditional VaR is somewhat better but left tail remains bad
- Gaussian conditional expected shortfall, which integrates over the left tail, would be terrible
- So we want more accurate assessment of things like $\text{VaR}_T^{p_{T+1|T}}$ than those obtained under Gaussian assumptions
  - Doing so for all values of $p \in [0, 1]$ requires estimating the entire conditional return distribution
  - More generally, best-practice risk measurement is about tracking the entire conditional return distribution
Assume:

\[ r_{T+1} = \sigma_{T+1}/T \varepsilon_{T+1} \]

\[ \varepsilon_{T+1} \sim iid(0, 1) \]

Multiply \( \varepsilon_{T+1} \) draws by \( \sigma_{T+1}/T \) (fixed across draws, from a GARCH model) to build up the conditional density of \( r_{T+1} \).

- \( \varepsilon_{T+1} \) simulated from standard normal
- \( \varepsilon_{T+1} \) simulated from standard t
- \( \varepsilon_{T+1} \) simulated from kernel density fit to \( \frac{r_{T+1}}{\sigma_{T+1}/T} \)
- \( \varepsilon_{T+1} \) simulated from any density that can be simulated
Parameter-Driven Density Forecasting
Using $r = \sigma \varepsilon$ and SV

Assume:

$$r_{T+1} = \sigma_{T+1} \varepsilon_{T+1}$$

$$\varepsilon_{T+1} \sim iid(0, 1)$$

Multiply $\varepsilon_{T+1}$ draws by $\sigma_{T+1}$ draws (from a simulated SV model) to build up the conditional density of $r_{T+1}$.

– Again, $\varepsilon_{T+1}$ simulated from any density deemed relevant
Modeling Entire Return Distributions: Returns Standardized by RV are Approximately Gaussian

Figure: QQ Plot of S&P500 Returns Standardized by Realized Volatilities. We show quantiles of daily S&P500 returns standardized by \( \text{AvgRV} \) against the corresponding quantiles of a standard normal distribution. The sample period is January 2, 1990 through December 31, 2010. The units on each axis are standard deviations.
A Special Parameter-Driven Density Forecasting Approach
Using \( r = \sigma \varepsilon \) and RV
(Log-Normal / Normal Mixture)

Assume:

\[
r_{T+1} = \sigma_{T+1} \varepsilon_{T+1}
\]

\[
\varepsilon_{T+1} \sim iid(0, 1)
\]

Multiply \( \varepsilon_{T+1} \) draws from \( N(0, 1) \) by \( \sigma_{T+1} \) draws (from a simulated RV model fit to log realized standard deviation) to build up the conditional density of \( r_{T+1} \).
Pitfalls of the “$r = \sigma \varepsilon$” Approach

In the conditionally Gaussian case we can write with no loss of generality:

\[ r_{T+1} = \sigma_{T+1}/T \varepsilon_{T+1} \]
\[ \varepsilon_{T+1} \sim iidN(0, 1) \]

But in the conditionally non-Gaussian case there is potential loss of generality in writing:

\[ r_{T+1} = \sigma_{T+1}/T \varepsilon_{T+1} \]
\[ \varepsilon_{T+1} \sim iid(0, 1), \]

because there may be time variation in conditional moments other than $\sigma_{T+1}/T$, and using $\varepsilon_{T+1} \sim iid(0, 1)$ assumes that away
Rigorous Modeling III

Conditional Asset-Level (Multivariate) Volatility Dynamics from “Daily” Data
Multivariate volatility models useful for portfolio-level risk measurement (VaR, ES, etc.)

But what about risk management questions:

- Portfolio risk change under a certain scenario involving price movements of set of assets or asset classes?
- Portfolio risk change if certain correlations increase suddenly
- Portfolio risk change if I double my holdings of Intel?
- How do optimal portfolio shares change if the covariance matrix moves in a certain way?

Similarly, what about almost any other question in asset pricing, hedging, trading? Almost all involve correlation.
Basic Framework and Issues I

\[ N \times 1 \text{ return vector } R_t \]

\[ N \times N \text{ covariance matrix } \Omega_t \]

- \( \frac{N(N+1)}{2} \) distinct elements
- Structure needed for pd or even psd
- Huge number of parameters even for moderate \( N \)
- And \( N \) may be not be moderate!
Basic Framework and Issues II

Univariate:

\[ r_t = \sigma_t z_t \]

\[ z_t \sim i.i.d.(0, 1) \]

Multivariate:

\[ R_t = \Omega_t^{1/2} Z_t \]

\[ Z_t \sim i.i.d.(0, I) \]

where \( \Omega_t^{1/2} \) is a “square-root” (e.g., Cholesky factor) of \( \Omega_t \).
Ad Hoc Exponential Smoothing (RM)

\[ \Omega_t = \lambda \Omega_{t-1} + (1 - \lambda) R_{t-1} R'_{t-1} \]

- Assumes that the dynamics of all the variances and covariances are driven by a single scalar parameter \( \lambda \) (identical smoothness)
- Guarantees that the smoothed covariance matrices are pd so long as \( \Omega_0 \) is pd
- Common strategy is to set \( \Omega_0 \) equal to the sample covariance matrix \( \frac{1}{T} \sum_{t=1}^{T} R_t R'_t \) (which is pd if \( T > N \))
- But covariance matrix forecasts inherit the implausible scaling properties of the univariate RM forecasts and will in general be suboptimal
Multivariate GARCH(1,1)

\[ \text{vech}(\Omega_t) = \text{vech}(C) + B \text{vech}(\Omega_{t-1}) + A \text{vech}(R_{t-1}R'_{t-1}) \]

- \text{vech} operator converts the upper triangle of a symmetric matrix into a \( \frac{1}{2}N(N+1) \times 1 \) column vector

- \( A \) and \( B \) matrices are both of dimension \( \frac{1}{2}N(N+1) \times \frac{1}{2}N(N+1) \)

- Even in this “parsimonious” GARCH(1,1) there are \( O(N^4) \) parameters
  - More than 50 million parameters for  \( N = 100 \)!
Encouraging Parsimony: Diagonal GARCH(1,1)

Diagonal GARCH constrains $A$ and $B$ matrices to be diagonal.

\[
vech(\Omega_t) = vech(C) + (I\beta) vech(\Omega_{t-1}) + (I\alpha) vech(R_{t-1}R_{t-1}')
\]

– Still $O(N^2)$ parameters.
Encouraging Parsimony: Scalar GARCH(1,1)

Scalar GARCH constrains $A$ and $B$ matrices to be scalar:

\[ \text{vech}(\Omega_t) = \text{vech}(C) + (I \beta) \text{vech}(\Omega_{t-1}) + (I \alpha) \text{vech}(R_{t-1}R'_{t-1}) \]

– Mirrors RM, but with the important difference that the $\Omega_t$ forecasts now revert to $\Omega = (1 - \alpha - \beta)^{-1}C$

– Fewer parameters than diagonal, but still $O(N)^2$ (because of $C$)
Encouraging Parsimony: Covariance Targeting

Recall variance targeting:

\[
\hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^{T} r_t^2, \quad \sigma^2 = \frac{\omega}{1 - \alpha - \beta} \implies \text{take } \omega = (1 - \alpha - \beta)\hat{\sigma}^2
\]

Covariance targeting is the obvious multivariate generalization:

\[
vech(C) = (I - A - B) \ vech\left( \frac{1}{T} \sum_{t=1}^{T} R_t R_t' \right)
\]

– Encourages both parsimony and reasonableness
Constant Conditional Correlation (CCC) Model

[Key is to recognize that correlation matrix is the covariance matrix of standardized returns]

Two-step estimation:

- Estimate $N$ appropriate univariate GARCH models
- Calculate standardized return vector, $\hat{e}_t = R_t \hat{D}_t^{-1}$
- Estimate correlation matrix $\Gamma$ (assumed constant) as
  \[ \frac{1}{T} \sum_{t=1}^{T} \hat{e}_t \hat{e}_t' \]
  - Quite flexible as the $N$ models can differ across returns
Dynamic Conditional Correlation (DCC) Model

Two-step estimation:

- Estimate $N$ appropriate univariate GARCH models
- Calculate standardized return vector, $\hat{e}_t = R_t \hat{D}_t^{-1}$
- Estimate correlation matrix $\Gamma_t$ (assumed to have scalar GARCH(1,1)-style dynamics) as following

$$ vech(\Gamma_t) = vech(C) + (I\beta)vech(\Gamma_{t-1}) + (I\alpha)vech(e_{t-1}e'_{t-1}) $$

- “Correlation targeting” is helpful
Time-varying correlations assumed identical across all pairs of assets, which implies:

\[ \Gamma_t = (1 - \rho_t) \mathcal{I} + \rho_t \mathcal{J}, \]

where \( \mathcal{J} \) is an \( N \times N \) matrix of ones.

Analytical inverse facilitates estimation:

\[ \Gamma_t^{-1} = \frac{1}{(1 - \rho_t)} \left[ \mathcal{I} - \frac{\rho_t}{1 + (N - 1)\rho_t} \mathcal{J} \right] \]

Assume GARCH(1,1)-style conditional correlation structure:

\[ \rho_t = \omega + \alpha u_t + \beta \rho_{t-1} \]

Updating rule is naturally given by the average conditional correlation of the standardized returns,

\[ u_t = \frac{2 \sum_{i=1}^{N} \sum_{j>i}^{N} e_{i,t} e_{j,t}}{N \sum_{i=1}^{N} e_{i,t}^2} \]

Three parameters, \( \omega, \alpha \) and \( \beta \), to be estimated.
Figure: Time-Varying International Equity Correlations. The figure shows the estimated equicorrelations from a DECO model for the aggregate equity index returns for 16 different developed markets from 1973 through 2009.
Factor Structure

\[ R_t = \lambda F_t + \nu_t \]

where

\[ F_t = \Omega_{Ft}^{1/2} Z_t \]

\[ Z_t \sim i.i.d.(0, \ I) \]

\[ \nu_t \sim i.i.d.(0, \ \Omega_{\nu}) \]

\[ \Rightarrow \Omega_t = \lambda \Omega_{Ft} \lambda' + \Omega_{\nu t} \]
One-Factor Case with Everything Orthogonal

\[ R_t = \lambda f_t + \nu_t \]

where

\[ f_t = \sigma_{ft} z_t \]

\[ z_t \sim i.i.d.(0, 1) \]

\[ \nu_t \sim i.i.d.(0, \sigma^2_\nu) \]

\[ \Omega_t = \sigma^2_{ft} \lambda \lambda' + \sigma^2_\nu \]

\[ \sigma^2_{it} = \sigma^2_{ft} \lambda_i^2 + \sigma^2_\nu \]

\[ \sigma^2_{ijt} = \sigma^2_{ft} \lambda_i \lambda_j \]
Rigorous Modeling IV

Conditional Asset-Level (Multivariate) Volatility Dynamics from High-Frequency Data
Realized Covariance

\[ dP(t) = M(t) \, dt + \Omega(t)^{1/2} \, dW(t) \]

\[ RCov_t(\Delta) \equiv \sum_{j=1}^{N(\Delta)} R_{t-1+j\Delta,\Delta} R'_{t-1+j\Delta,\Delta} \]

\[ RCov_t(\Delta) \rightarrow ICov_t = \int_{t-1}^{t} \Omega(\tau) \, d\tau \]

- p.d. so long as \( N(\Delta) > N \); else use regularization methods
Asynchronous Trading and the Epps Effect

- Epps effect biases covariance estimates downward

- Can overcome Epps by lowering sampling frequency to accommodate least-frequently-traded asset, but that wastes data

- Opposite extreme: Calculate each pairwise realized covariance matrix using appropriate sampling; then assemble and regularize
Regularization (Shrinkage)

\[ \hat{\Omega}_t^S = \kappa \text{RCov}_t(\Delta) + (1 - \kappa) \gamma_t \]

- \( \gamma_t \) is p.d. and \( 0 < \kappa < 1 \)
- \( \gamma_t = I \) (naive benchmark)
- \( \gamma_t = \Omega \) (unconditional covariance matrix)
- \( \gamma_t = \sigma_M^2 b b' + D_\nu \) (one-factor market model)
Models for Realized Covariance Matrices

- Mimic univariate exponential smoothing:
  \[ \hat{\Omega}_{t+1|t} = \lambda \hat{\Omega}_{t|t-1} + (1 - \lambda) \hat{\Omega}_t \]

- Mimic scalar diagonal GARCH with multivariate regression:
  \[ \text{vech}(\hat{\Omega}_{t+1}) = \text{vech}(C) + \beta \text{vech}(\hat{\Omega}_t) + \xi_{t+1} \]

- Mimic DCC:
  \[ \text{vech}(Q_t) = \text{vech}(C) + \beta \text{vech}(Q_{t-1}) + \xi_t \]

- Maintain p.d. using Cholesky or matrix log
Multivariate GARCH-RV

\[ \text{vech}(\Omega_t) = \text{vech}(C) + B \ \text{vech}(\Omega_{t-1}) + A \ \text{vech}(\hat{\Omega}_{t-1}) \]

- Fine for 1-step
- Multi-step requires “closing the system” with an RV equation
  - Noureldin et al. (2011), multivariate HEAVY
Multivariate Return Distributions

– If reliable realized covariances are available, one could do a multivariate analog of the earlier lognormal/normal mixture model. But the literature thus far has focused primarily on conditional distributions for “daily” data.

Return version:

\[ Z_t = \Omega_t^{-1/2} R_t, \quad Z_t \sim i.i.d., \quad E_{t-1}(Z_t) = 0 \quad \text{Var}_{t-1}(Z_t) = I \]

Standardized return version (as in DCC):

\[ e_t = D_t^{-1} R_t, \quad E_{t-1}(e_t) = 0, \quad \text{Var}_{t-1}(e_t) = \Gamma_t \]

where \( D_t \) denotes the diagonal matrix of conditional standard deviations for each of the assets, and \( \Gamma_t \) refers to the potentially time-varying conditional correlation matrix.
Leading Examples

Multivariate normal:

\[ f(e_t) = C(\Gamma_t) \exp \left( -\frac{1}{2} e_t' \Gamma_t^{-1} e_t \right) \]

Multivariate \( t \):

\[ f(e_t) = C(d, \Gamma_t) \left( 1 + \frac{e_t' \Gamma_t^{-1} e_t}{(d-2)} \right)^{-(d+N)/2} \]

Multivariate asymmetric \( t \):

\[ f(e_t) = \frac{C(d, \Gamma_t) K_{d+N}^{d+N}}{2 \left( 1 + \frac{(e_t - \mu)' \Gamma_t^{-1} (e_t - \mu)}{d} \right)^{-(d+N)/2}} \left( \sqrt{d + (e_t - \mu)' \Gamma_t^{-1} (e_t - \mu)} \xi' \Gamma_t^{-1} \xi \right) \exp \left( (e_t - \mu)' \Gamma_t^{-1} (e_t - \mu) \right) \]

- More flexible than symmetric \( t \) but requires estimation of \( N \) asymmetry parameters simultaneously with the other parameters, which is challenging in high dimensions.

Copula methods sometimes provide a simpler two-step approach.
Copula Methods

Sklar’s Theorem:

\[ F(e) = G(F_1(e_1), \ldots, F_N(e_N)) \equiv G(u_1, \ldots, u_N) \equiv G(u) \]

\[ f(e) = \frac{\partial^N G(F_1(e_1), \ldots, F_N(e_N))}{\partial e_1 \ldots \partial e_N} = g(u) \times \prod_{i=1}^{N} f_i(e_i) \]

\[ \Rightarrow \log L = \sum_{t=1}^{T} \log g(u_t) + \sum_{t=1}^{T} \sum_{i=1}^{N} \log f_i(e_{i,t}) \]
Standard Copulas

Normal:

\[ g(u_t; \Gamma^*_t) = |\Gamma^*_t|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \Phi^{-1}(u_t)'(\Gamma^*_t^{-1} - I)\Phi^{-1}(u_t) \right\} \]

where \( \Phi^{-1}(u_t) \) refers to the \( N \times 1 \) vector of standard inverse univariate normals, and the correlation matrix \( \Gamma^*_t \) pertains to the \( N \times 1 \) vector \( e^*_t \) with typical element,

\[ e^*_{i,t} = \Phi^{-1}(u_{i,t}) = \Phi^{-1}(F_i(e_{i,t})). \]

– Often does not allow for sufficient dependence between tail events.

– t copula

– Asymmetric t copula
Multivariate Distribution Simulation (General Case)

Simulate using:

\[ R_t = \hat{\Omega}_t^{1/2} Z_t \]

\[ Z_t \sim i.i.d. (0, I) \]

- \( Z_t \) may be drawn from parametrically-(Gaussian, t, ...) or nonparametrically-fitted distributions, or with replacement from the empirical distribution.
Multivariate Distribution Simulation (Factor Case)

Simulate using:

\[ R_{F,t} = \hat{\Omega}_{F,t}^{1/2} Z_{F,t} \]

\[ R_t = \hat{B} R_{F,t} + \nu_t \]

- \( Z_{F,t} \) and \( \nu_t \) may be drawn from parametrically- or nonparametrically-fitted distributions, or with replacement from the empirical distribution.
Figure: Average Threshold Correlations for Sixteen Developed Equity Markets. The solid line shows the average empirical threshold correlation for GARCH residuals across sixteen developed equity markets. The dashed line shows the threshold correlations implied by a multivariate standard normal distribution with constant correlation. The line with square markers shows the threshold correlations from a DECO model estimated on the GARCH residuals from the 16 equity markets. The figure is based on weekly returns from 1973 to 2009.
Rigorous Modeling V

Measuring Connectedness in the Small and in the Large
Financial and Economic Connectedness (Diebold-Yilmaz)

- Market Risk, Portfolio Concentration Risk (return connectedness)
- Credit Risk (default connectedness)
- Counterparty Risk, Gridlock Risk (bilateral and multilateral contractual connectedness)
- Systemic Risk (system-wide connectedness)
- Business Cycle Risk (local or global real output connectedness)
Covariance and Correlation

- So pairwise...
- So linear...
- So Gaussian...
We Will Take a Different Approach…

– Natural framework with direct motivation

– Firmly grounded in network theory

– Track transmissions and receipts, from highly granular to highly aggregated
Two Natural Questions

A natural modeling question:
What fraction of the $H$-step-ahead prediction-error variance of variable $i$ is due to shocks in variable $j$, $\forall i, j$?

Variance decomposition: $d_{ij}^H, \forall i, j$

A natural financial/economic connectedness question:
What fraction of the $H$-step-ahead prediction-error variance of variable $i$ is due to shocks in variable $j$, $\forall j \neq i$?

Non-own elements of the variance decomposition: $d_{ij}^H, \forall j \neq i$
### N-Variable Connectedness Table

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\ldots$</th>
<th>$x_N$</th>
<th>From Others to $i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$d_{11}^H$</td>
<td>$d_{12}^H$</td>
<td>$\ldots$</td>
<td>$d_{1N}^H$</td>
<td>$\Sigma_{j=1}^N d_{1j}^H, j \neq 1$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$d_{21}^H$</td>
<td>$d_{22}^H$</td>
<td>$\ldots$</td>
<td>$d_{2N}^H$</td>
<td>$\Sigma_{j=1}^N d_{2j}^H, j \neq 2$</td>
</tr>
<tr>
<td>$\vdots$</td>
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<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$x_N$</td>
<td>$d_{N1}^H$</td>
<td>$d_{N2}^H$</td>
<td>$\ldots$</td>
<td>$d_{NN}^H$</td>
<td>$\Sigma_{j=1}^N d_{Nj}^H, j \neq N$</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>From Others</th>
<th>$\Sigma_{i=1}^N d_{i1}^H$</th>
<th>$\Sigma_{i=1}^N d_{i2}^H$</th>
<th>$\ldots$</th>
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<th>$\Sigma_{i,j=1}^N d_{ij}^H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>From $j$</td>
<td>$i \neq 1$</td>
<td>$i \neq 2$</td>
<td>$i \neq N$</td>
<td>$i \neq j$</td>
<td>$i \neq j$</td>
</tr>
</tbody>
</table>

Upper-left block is variance decomposition matrix, $D$

Connectedness involves the **non-diagonal** elements of $D$.  

---

*Penn University of Pennsylvania*
Connectedness Measures

- Pairwise Directional: $C^H_{i \leftarrow j} = d^H_{ij}$ ("$i$’s imports from $j$")
- Net: $C^H_{ij} = C^H_{j \leftarrow i} - C^H_{i \leftarrow j}$ ("$ij$ bilateral trade balance")

Total Directional:

- From others to $i$: $C^H_{i \leftarrow \bullet} = \sum_{j=1 \atop j \neq i}^{N} d^H_{ij}$ ("$i$’s total imports")
- To others from $j$: $C^H_{\bullet \leftarrow j} = \sum_{i=1 \atop i \neq j}^{N} d^H_{ij}$ ("$j$’s total exports")
- Net: $C^H_{i} = C^H_{\bullet \leftarrow i} - C^H_{i \leftarrow \bullet}$ ("$i$’s multilateral trade balance")

Total: $C^H = \frac{1}{N} \sum_{i,j=1 \atop i \neq j}^{N} d^H_{ij}$ ("total world exports")
Networks I: Representation

\[
A = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}
\]

Adjacency Matrix (Symmetric)

\( A_{ij} = 1 \) if nodes \( i, j \) linked

\( A_{ij} = 0 \) otherwise
Networks I: Degree

Degree of node $i$, $d_i$:

$$d_i = \sum_{j=1}^{N} A_{ij}$$

Discrete degree distribution, $P(d)$, on $0, \ldots, N - 1$

Mean degree, $E(d)$, is the key connectedness measure

Beautiful results (e.g., “small world”) involve the mean degree:

$$\text{diameter} \approx \frac{\ln N}{\ln E(d)}$$
Networks II: Representation
(Weighted, Directed)

\[ A = \begin{pmatrix}
0 & .5 & .7 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & .3 & 0 \\
0 & 0 & 0 & .7 & 0 & .3 \\
.3 & .5 & 0 & 0 & 0 & 0 \\
.5 & 0 & 0 & 0 & 0 & .3 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \]

“to i, from j”
Networks II: Degree
(Weighted, Directed)

\( A_{ij} \in [0, 1] \) depending on connection strength

Two degrees:

\[
d_{i}^{\text{from}} = \sum_{j=1}^{N} A_{ij}
\]

\[
d_{j}^{\text{to}} = \sum_{i=1}^{N} A_{ij}
\]

Continuous “from” and “to” degree distributions on \([0, N - 1]\)

Mean degree \( E(d) \) remains the key connectedness measure
Connectedness Table

<table>
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<tr>
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<td>...</td>
<td>$d_{NN}^H$</td>
<td>$\sum_{j\neq N} d_{Nj}^H$</td>
</tr>
</tbody>
</table>

To Others
\[ \sum_{i\neq1} d_{i1}^H \quad \sum_{i\neq2} d_{i2}^H \quad \ldots \quad \sum_{i\neq N} d_{iN}^H \quad \sum_{i\neq j} d_{ij}^H \]

$C_{i\leftarrow \bullet}^H = \sum_{j=1}^{N} d_{ij}^H$, are the “from degrees”

$C_{\bullet\leftarrow j}^H = \sum_{i=1}^{N} d_{ij}^H$, are the “to degrees”

$C^H = \frac{1}{N} \sum_{i,j=1}^{N} d_{ij}^H$, is the mean degree (to or from)
Relationships to other market-based measures?

- Marginal expected shortfall
- Expected capital shortfall
- CoVaR
- $\Delta$CoVaR
MES and ECS

\[ \text{MES}^{j|mkt}_{T+1|T} = E_T [r_j, T+1|C (r_{mkt}, T+1)] \]

- Sensitivity of firm j’s return to extreme market event \( C \)
- Market-based “stress test” of firm j’s fragility
- Like “total directional connectedness from” (from degree)

\[ \text{ECS}^{j|mkt}_{T+1|T} = a_{0j} + a_{1j} \text{MES}^{j|mkt}_{T+1|T} \]

- \( a_{0j} \) depends on firm j’s “prudential ratio” of asset value to equity as well as its debt composition
- \( a_{1j} \) depends on firm j’s prudential ratio and initial capital
CoVaR and $\Delta$CoVaR

VaR: $p = \Pr_T \left( r_{j,T+1} < -\text{VaR}^{p,j}_{T+1|T} \right)$

CoVaR: $p = \Pr_T \left( r_{j,T+1} < -\text{CoVaR}^{j|i}_{T+1|T} \mid \mathbb{C}(r_{i,T+1}) \right)$

Marketwide: $p = \Pr_T \left( r_{mkt,T+1} < -\text{CoVaR}^{mkt|i}_{T+1|T} \mid \mathbb{C}(r_{i,T+1}) \right)$

- Measures tail-event linkages
- Leading choice of $\mathbb{C}(r_{i,T+1})$ is that firm $i$ breaches its VaR
- Like “total directional connectedness to” (to degree)

Delta CoVaR: $\Delta \text{CoVaR}^{j|i}_{T+1|T} = \text{CoVaR}^{j|\text{VaR}(i)}_{T+1|T} - \text{CoVaR}^{j|\text{Med}(i)}_{T+1|T}$

Marketwide: $\Delta \text{CoVaR}^{mkt|i}_{T+1|T} = \text{CoVaR}^{mkt|\text{VaR}(i)}_{T+1|T} - \text{CoVaR}^{mkt|\text{Med}(i)}_{T+1|T}$
Estimating Connectedness

Thus far we’ve worked under correct specification, in population:

$$C(x, H, B(L))$$

Now we want:

$$\hat{C} \left( x, H, B(L), M(L; \hat{\theta}) \right),$$

and similarly for other variants of connectedness
Many Interesting Issues

- x objects: Returns? **Return volatilities**? Real activities?
- x universe: How many and which ones?
  (\( \approx 15 \text{ major financial institutions} \))
- x frequency: **Daily**? Monthly? Quarterly?

- \( H: \) Match VaR horizon? Holding period?

- \( M: \) **VAR**? Structural?

- Identification of variance decompositions:
  Cholesky? **Generalized**? Structural?

- Estimation: **Classical**? Bayesian?
Connectedness of Major U.S. Financial Institutions

\[ \hat{C}(x, H, B(L), M(L; \hat{\theta})) \]

- **\( x \):** Thirteen daily realized stock return volatilities
  - Commercial banks: JP Morgan Chase (JPM), Bank of America (BAC), CitiGroup (C), Wells Fargo (WFC), Bank of New York Mellon (BK), U.S. BankCorp (USB), PNC Bank (PNC)
  - Investment Banks: Goldman Sachs (GS), Morgan Stanley (MS)
  - GSEs: Fannie Mae (FNM), Freddie Mac (FRE)
  - Insurance: AIG (AIG)
  - Specialized: American Express (AXP)

- **\( H \):** 12 days

- **\( M(L; \theta) \):** logarithmic VAR(3), generalized identification, 5/4/1999 - 4/30/2010
## Full-Sample Connectedness Table

<table>
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<tr>
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Estimating Time-Varying Connectedness

Before:
\[ C(x, H, B(L), M(L; \theta)) \]
\[ \hat{C}(x, H, B(L), M(L; \hat{\theta})) \]

Now:
\[ C_t(x, H, B_t(L), M(L; \theta_t)) \]
\[ \hat{C}_t(x, H, B_t(L), M(L; \hat{\theta}_t)) \]

- Time-varying parameters: **Rolling estimation**? Smooth TVP model? Regime-switching?

(100-day estimation window)
Rolling Total Connectedness
Net Pairwise Directional Connectedness, The Lehman Bankruptcy

Figure: Sept. 16, 2008
In Conclusion...
Epilogue

- Reliable risk measurement requires *conditional* models that allow for time-varying volatility.
- Risk measurement may be done using univariate density models directly for portfolio returns. Many important recent developments.
- Certain aspects of risk *management* require fully-specified multivariate models. Many important recent developments, especially for $N$ large.
- Volatility measures based on high-frequency return data hold great promise. A base-asset factor approach is often useful.
- The business cycle emerges as a key macroeconomic fundamental driving risk. Hence conditioning also applies at longer horizons.