Minimal Supersolutions of Backward Stochastic Differential Equations and Robust Hedging

SAMUEL DRAPEAU
Humboldt-Universität zu Berlin

BSDEs, Numerics and Finance
Oxford
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joint work with GREGOR HEYNE and MICHAEL KUPPER
Supersolutions of BSDE
Motivation

Superhedging problem

\[ Y_0 + \int_0^T Z_u dS_u \geq \xi \]

- \( Y_0 \): superhedging price of \( \xi \)
- \( Z \): superhedging strategy
Motivation

Superhedging problem

\[ Y_t + \int_t^T Z_u dS_u \geq \xi \]

- \( Y_t \): superhedging price of \( \xi \) at \( t \)
- \( Z \): superhedging strategy

Trading gains

Value

\[ 0 \quad 1 \quad 2 \quad \text{superhedging prices at time } t \]

Minimal superhedging price at time \( t \)

\[ \xi \]

Time
Supersolutions of Backward Stochastic Differential Equations

Motivation

Definition

\[(Y, Z)\] is a supersolution of the Backward Stochastic Differential Equation with driver \(g\) and terminal condition \(\xi\) if

\[Y_t - \int_t^T g(Y_u, Z_u)du + \int_t^T Z_u dW_u \geq \xi \quad \forall t \in [0, T]\]

\(Y\) = value process
\(Z\) = control process

Equality instead of inequality: \((Y, Z)\) is solution of the BSDE.

Extensively studied: Bismut, Pardoux, Peng, Ma, Protter, Yong, Briand, Hu, Kobylanski, Touzi, Delbaen, Imkeller, El Karoui, ...

Applications in utility maximization, stochastic games, stochastic equilibria, ...
Supersolutions are typically not unique.

Find a **minimal supersolution** \((Y_{\text{min}}, Z_{\text{min}})\)!

That is \(Y_{\text{min}} \leq Y\) for any other supersolution \((Y, Z)\).
Filtrated probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\), filtration generated by a Brownian motion \(W\) satisfying the usual conditions.

\[
\begin{cases}
Y_s - \int_s^t g(Y_u, Z_u) du + \int_s^t Z_u dW_u \geq Y_t, & 0 \leq s \leq t \leq T \\
Y_T \geq \xi
\end{cases}
\]

\(0.1\)

1. \(\xi\) is \(\mathcal{F}_T\)-measurable.
2. \(Y\) is \((\mathcal{F}_t)\)-adapted and càdlàg \(\sim S\).
3. \(Z\) is \((\mathcal{F}_t)\)-progressive, such that \(\int_0^T Z_u^2 du < +\infty\) and !
Minimal Supersolutions

Filtrated probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\), filtration generated by a Brownian motion \(W\) satisfying the usual conditions.

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1. \(\xi\) is \(\mathcal{F}_T\)-measurable.
2. \(Y\) is \((\mathcal{F}_t)\)-adapted and càdlàg \(\sim \mathcal{S}\).
3. \(Z\) is \((\mathcal{F}_t)\)-progressive, such that \(\int_0^T Z_u^2 \, du < +\infty\) and \(Z\) is admissible, i.e. \(\int Z \, dW\) is a supermartingale (\(\rightarrow\) Dudley and Harrison/Pliska) \(\sim \mathcal{L}\).

The set of supersolutions with driver \(g\) and terminal condition \(\xi\)

\[
\mathcal{A} := \{(Y, Z) \in \mathcal{S} \times \mathcal{L} : (Y, Z) \text{ fulfills } (0.1)\}
\]
A generator is a lower semicontinuous function

\[ g : \mathbb{R} \times \mathbb{R}^d \rightarrow ] - \infty, \infty]. \]

Additional properties:

(\textbf{Pos}) \quad g(y, z) \in [0, +\infty] \text{ for all } (y, z).

(\textbf{Conv}) \quad z \mapsto g(y, z) \text{ is convex.}

(\textbf{Mon}) \quad g(y, z) \geq g(y', z) \text{ for all } y \geq y'.

(\textbf{Mon'}) \quad g(y, z) \leq g(y', z) \text{ for all } y \geq y'.
A natural candidate for the value process of a minimal supersolution:

\[ \hat{\mathcal{E}}_t = \text{ess inf} \{ Y_t : (Y, Z) \in \mathcal{A} \}, \quad t \in [0, T] \]

Question: Does there exist a càdlàg modification $\mathcal{E}$ of $\hat{\mathcal{E}}$ and a control process $Z \in \mathcal{L}$ such that $(\mathcal{E}, Z)$ is a supersolution?
Minimal Supersolutions

A natural candidate for the value process of a minimal supersolution:

$$\hat{E}_t = \text{ess inf} \{ Y_t : (Y, Z) \in A \}, \quad t \in [0, T]$$

**Theorem:**

Assume \((\text{Pos})\), \((\text{Conv})\) and either \((\text{Mon})\) or \((\text{Mon}')\). Suppose \(\xi^- \in L^1\) and \(A \neq \emptyset\). Then

$$E_t := \hat{E}_t^+ = \lim_{s \downarrow t, s \in Q} \hat{E}_s$$

is the value process of the unique minimal supersolution, that is, there exists a unique control process \(Z\) such that \((E, Z) \in A\).

- **Compactness** (Delbaen and Schachermayer) versus fixpoint.
- Drop positivity for \((\text{Pos}')\) \(g(y, z) \geq az + b\). (utility maximization)
- Gregor Heyne, Michael Kupper and Christoph Mainberger, drop convexity in \(z\) for \(g(y, 0) = 0\). (Barlow and Protter).
Any sequence \((x_n)\) in \(\mathbb{R}^d\) such that \(\sup_{n \in \mathbb{N}} \|x_n\| < \infty\) has a subsequence \((x_{n_k})\) converging to some \(x \in \mathbb{R}^d\).

Let \((X_n)\) be a sequence of random variables in \(L^2(\Omega, \mathcal{F}, P)\) such that \(\sup_{n \in \mathbb{N}} E[X_n^2] < \infty\). Then there exists a sequence \(Y_n \in \text{conv}(X_n, X_{n+1}, \ldots)\) such that \(Y_n \to Y\) in \(L^2(\Omega, \mathcal{F}, P)\).

(Delbaen/Schachermayer) Let \(\int H^n dW\) be a \(\mathcal{H}^1\)-bounded sequence of martingales. Then there exist \(K^n \in \text{conv}\{H^n, H^{n+1}, \ldots\}\) and a localizing sequence of stopping times \((\tau^n)\) such that \(\int K^n dW |_{\tau^n} \to \int K dW\) in \(\mathcal{H}^1\).
1) Paste strategies between stopping times \( \sim \) construct \((Y^n, Z^n) \subset A\) with,
\[
\hat{E}^n_t \geq Y^n_t - 1/n, \quad \text{and} \quad Y^n_t \geq Y^{n+1}_t,
\]

2) \( Y = \lim_{n \to \infty} Y^n \) and \( \hat{E} \) are supermartingales \( \sim \) \( E := \hat{E}^+ = Y^+ \).

3) Show that \( \hat{E}_t \geq E_t \).

4) There is a localizing sequence \((\sigma_k)\) such that
\[
\left( \int Z^n dW \right)^{\sigma_k}
\]
is bounded in \( \mathcal{H}^1 \).

5) DELBAEN and SCHACHERMAYER \( \sim \) convex combinations such that
\[
\int_0^t \tilde{Z}_s^n dW_s \overset{n \to +\infty}{\longrightarrow} \int_0^t Z_s dW_s.
\]

6) Verification with \((E, Z)\) is based on Helly’s theorem and Fatou’s lemma.
Minimal Supersolutions

⇔

Nonlinear Expectations

(∼ Peng’s $g$-expectations)
Stability results

For any “nice” generator $g$ the mapping

$$\mathcal{E}^g : \xi \mapsto \text{minimal supersolution with terminal condition } \xi$$

satisfies

<table>
<thead>
<tr>
<th>Condition</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N)</td>
<td>$\mathcal{E}_0^g (m) = m$</td>
</tr>
<tr>
<td>(T)</td>
<td>$\mathcal{E}_0^g (\xi + m) = \mathcal{E}_0^g (\xi) + m$</td>
</tr>
<tr>
<td>(TC)</td>
<td>$\mathcal{E}_0^g (\xi) = \mathcal{E}_0^g (\mathcal{E}_t^g (\xi))$</td>
</tr>
<tr>
<td>Linearity:</td>
<td>$\implies$ nonlinearity expectation</td>
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$$E[\xi] = \int_{\Omega} \xi(\omega) P(d\omega)$$

$$E[m] = m$$

$$E[\xi + m] = E[\xi] + m$$

$$E[\xi] = E [E[\xi | \mathcal{F}_t]]$$

$$E[\lambda \xi^1 + \xi^2] = \lambda E[\xi^1] + E[\xi^2]$$

$\sim$ nonlinear expectation
The nonlinear expectation $\mathcal{E}_0^g(\cdot)$ satisfies

- **Monotone convergence:**
  
  $0 \leq \xi^n \uparrow \xi$ implies $\mathcal{E}_0^g(\xi) = \lim_n \mathcal{E}_0^g(\xi^n)$

- **Fatou’s lemma:** $\mathcal{E}_0^g(\lim \inf_n \xi^n) \leq \lim \inf_n \mathcal{E}_0^g(\xi^n)$

- is $\sigma(L^1, L^\infty)$-lower semicontinuous.

If $g$ is independent of $y$, by convex duality:

$$
\mathcal{E}_0^g(\xi) = \sup_{Q \ll P} \{ E_Q[\xi] - \alpha_{\min}(Q) \}
$$

$=$ representation of a convex risk measure
Model Uncertainty and Robust Hedging

(〜 Peng’s $G$-expectation)
The role of the probability measure $P$

The probability $P$ defines the dynamics of the process

$P \leftrightarrow \text{probabilistic model, e.g. on } C([0, T]; \mathbb{R})$

What is the probability measure $P$?
The role of the probability measure $P$

$P$ can only partially be identified by statistical methods

\[\downarrow\]

Take into account a family $\mathcal{P}$ of probability measures (models)

\[\sim\text{ Model Uncertainty}\]

Remark: The probability measures are typically singular!

\[\mathcal{P}(A) := \sup_{P \in \mathcal{P}} P[A] = \text{capacity}\]

\[\sim\text{ Denis, Martini, Peng, Hu, Bion-Nadal, Soner, Touzi, Zhang, Nutz,\ldots}\]

E.g.

\[\frac{dS_t(\theta)}{S_t(\theta)} = \mu dt + \theta dW_t, \quad \underline{\theta} \leq \theta \leq \overline{\theta}\]
Minimal Supersolutions of Robust BSDEs

Setting

- Θ is a set of volatility processes:

\[ \theta : \Omega \times [0, T] \rightarrow \mathbb{R}^{++} \quad (\mathbb{S}^0). \]

- Our state space:

\[ \tilde{\Omega} := \Omega \times \Theta \]

- Driving process: \( \tilde{W} : \tilde{\Omega} \times [0, T] \rightarrow \mathbb{R} \), where

\[ \tilde{W}(\theta) = \int \theta dW, \quad \theta \in \Theta \]

- Progressively learning about the volatility

\[ \sim \quad \tilde{\mathcal{F}}_t := \sigma(\tilde{W}_s : s \leq t), \quad t \in [0, T] \]

In general not right-continuous.
Let $\mu^\theta$ be the probability measure induced by $\tilde{\mathcal{W}}(\theta)$ on $C([0, T], \mathbb{R}^d)$ with the Borel $\sigma$-algebra. These probability measures are singular to each others. There is no dominating probability measures!

$$\tilde{\mathcal{P}}[A] := \sup_{\theta \in \Theta} P[A(\theta)], \quad A \in \tilde{\mathcal{F}}_T$$

Properties (like equality and inequalities) holds quasi-surely if the event $B$, where they do not hold is a polar set, i.e., $B \in \tilde{\mathcal{F}}_T$ with $\tilde{\mathcal{P}}[B] = 0$.

We assume that $\{\mu^\theta : \theta \in \Theta\}$ is weakly compact.

$\rightarrow$ Denis, Martini, Peng, Hu, Bion-Nadal, Soner, Touzi, Zhang, Nutz
For all $\theta \in \Theta$,

$$
\begin{cases}
Y_\sigma(\theta) - \int_\sigma^T g_u(Y_u(\theta), Z_u(\theta))du + \int_\sigma^T Z_u(\theta)d\tilde{W}_u(\theta) \geq Y_\tau(\theta), \\
Y_T(\theta) \geq \xi(\theta)
\end{cases}
$$

(0.2)

where $\sigma, \tau$ are $(\mathcal{F}_t)$-stopping times with $0 \leq \sigma \leq \tau \leq T$.

1. $\xi$ is $\tilde{\mathcal{F}}_T$-measurable.
2. $Y$ is càdlàg and $Y_t \in L^1_b(\tilde{\mathcal{F}}_t) \cap C(\tilde{\mathcal{F}}_t)$, $Y(\theta)$ is optional $\sim \tilde{S}$
3. $Z$ is $(\tilde{\mathcal{F}}_t)$-predictable, such that $\int_0^T Z_u^2(\theta)\theta_u^2 du < +\infty$ and $\int Z(\theta)d\tilde{W}(\theta)$ is a supermartingale for all $\theta \in \Theta \sim \tilde{\mathcal{L}}$
4. $g : \mathbb{R} \times \mathbb{R} \to (-\infty, +\infty]$, such that $g(\theta)$ is a generator as before for all $\theta \in \Theta$.

The set of supersolutions with driver $g$ and terminal condition $\xi$

$$\mathcal{A} := \left\{ (Y, Z) \in \tilde{S} \times \tilde{\mathcal{L}} : (Y, Z) \text{ fulfills (0.2)} \right\}$$
As a usual, our natural candidate for the minimal supersolution

\[ \hat{E}_t = \text{ess inf} \{ Y_t : (Y, Z) \in \mathcal{A} \} \]

However, there is no reference probability measure!
\[ \rightarrow \text{Bion-Nadal, Nutz, ...} \]

We consider the infimum:

\[ \hat{E}_t = \inf \{ Y_t : (Y, Z) \in \mathcal{A} \} \]
Our existence Theorem reads as follows

**Theorem:**

Assume \((\text{Pos})\), \((\text{Conv})\) and either \((\text{Mon})\) or \((\text{Mon}')\). Suppose \(\xi^- \in L^1_b(\tilde{F}_T)\), \(A \neq \emptyset\) and \(\hat{E}_t \in C(\tilde{F}_t)\). Then there exists a làdlàg modification \(\mathcal{E}\) of \(\hat{E}\), which is the value process of the unique minimal supersolution, that is, there exists a unique control process \(Z\) such that

\[(\mathcal{E}, Z) \in A.
\]

We give conditions under which the \(\hat{E}\) fulfills these assumptions (Markovian Setting).
Thank You!